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OPTIMAL PORTFOLIO STRATEGIES IN THE PRESENCE OF REGIMES IN ASSET RETURNS APPLIED TO THE BRAZILIAN FINANCIAL MARKET

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2019

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Master's dissertation presented to the Instituto Coppead de Administração, Universidade Federal do Rio de Janeiro, as part of the mandatory requirements in order to obtain the title of Master in Business Administration (M.Sc.).

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Resumo

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Esta dissertação analisa estratégias ótimas de portfólio para um investidor brasileiro em uma economia com mudanças de regimes que possui quatro estados não observáveis. Aplicamos uma estratégia analítica aproximada para o problema dinâmico irrestrito utilizando os ativos financeiros mais importantes para um investidor brasileiro comum: dinheiro, renda fixa, ações domésticas e ações internacionais. Demonstramos que a aproximação é bastante precisa no cenário de quatro regimes e que a política de portfólio depende fortemente do estado corrente da economia, indicando que compensa trabalhar com múltiplos regimes. Como resultado, propomos uma estratégia de portfólio dinâmico em tempo real que apresenta performance superior ao ser comparada com as classes de ativos mais importantes do mercado. Esta estratégia pode ser uma solução realística para o problema de portfólio, que, de forma clara e direta, pode beneficiar os gestores de fundos de investimentos.

Palavras-chave: Brasil, solução realística, problema do investidor, riqueza terminal, estratégia ótima de portfólio, alocação dinâmica de ativos, tempo real, função de utilidade estocástica diferencial, classes de ativos, diversificação, prêmio de risco, múltiplos estados não-observáveis, economia com mudança de regimes, crash, bear, bull, recovery.

Abstract

Lewin, Marcelo. *Optimal portfolio strategies in the presence of regimes in asset returns applied to the Brazilian financial market.* 2019. 45 p. Dissertation (Master in Business Administration) – Universidade Federal do Rio de Janeiro – Instituto COPPEAD de Administração, Rio de Janeiro, 2019.

This dissertation analyzes optimal portfolio strategies for a Brazilian investor in a regime switching economy with four unobservable states. An approximate analytical solution is applied to the unconstrained dynamic problem using the most important financial assets for an ordinary Brazilian investor: money, fixed income, domestic stocks and international stocks. We demonstrate that the approximation is fairly accurate in the four-regime setting, and that the portfolio policy strongly depends on the current state of the economy, which indicates that the multiple regime framework pays off. As a consequence, we propose a real-time dynamic portfolio strategy that shows superior performance when compared to the most important asset classes of the market. This strategy can be a realistic solution for the portfolio problem, and fund managers could clear and directly benefit from it.

Keywords: Brazil, realistic solution, investor problem, terminal wealth, optimal portfolio strategy, dynamic asset allocation, real-time, stochastic differential utility function, asset classes, diversification, risk-premium, multiple unobservable states, regime switching economy, crash, bear, bull, recovery.

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1 Introduction

Much academic research has been applied to the study of asset allocation models since Markowitz (1952), Samuelson (1969) and Merton (1971) in order to attend the investors' request to maximize the total utility of future consumption and terminal wealth. Not only to solve the investor problem, but also in many other fields of finance, Hamilton (1989) has greatly contributed by introducing the Markov chains. This methodology is applied to capture shifts on the economy states, essential to predict cycles and the implicant change of parameters. The shifts between states are modelled by regimes that switch according to transition probabilities.

The importance of working with regime switching models in optimal portfolio allocation was confirmed by previous researches. For example, Ang and Bekaert (2004) presented the relevance of regime switching in tactical asset allocation, while Tu (2010) concluded that the certainty-equivalent losses originated from ignoring the regime switching in investment decisions are generally above 2% per year. On its turn, Guidolin and Hyde (2012) had concluded that the vector autoregressive models are inefficient to capture the regime shifts in asset returns.

Ang and Bekaert (2002) were the first to study portfolio allocation under a regime switching process. Applying a setting of two fully observable regimes, bull and bear markets, they worked with n risky assets and computed the utility coming from terminal wealth. With the same regime setting, Graflund and Nilsson (2003) used one risky asset and a riskless bond to analyze intertemporal hedging demands with power utility from terminal wealth. Both offered numerical solutions but did not include the possibility of consumption. Later, under two unobservable regimes, working with a single risky asset, Liu (2011) investigated the problem with consumption.

The literature has followed with Guidolin and Timmermann (2005, 2007), who identified the existence of four unobservable regimes in the series of stock and bond markets in the US economy. They considered consumption and power utility, and applied Monte Carlo techniques. Later, Campani and Garcia (2018) also worked under a setting of four unobservable regimes, including consumption. But in their study, felicity came from the stochastic differential utility function, which is one step ahead of the classical power utility function because it disentangles elasticity of intertemporal substitution (ψ) from relative risk aversion (γ). To solve the problem, these authors proposed an approximate analytical solution, which they claim to be fairly accurate.

This dissertation applies the solution of Campani and Garcia (2018), here named CG Model, and tests its accuracy in the Brazilian financial markets. We work with four unobservable regimes in the economy and solve the problem of a dynamic portfolio allocation in Brazil. This is the first time that the multiple regime framework is applied in Brazil for this purpose. Based on this setting, we present a real-time portfolio strategy and also demonstrate that the results obtained through this application of the CG model outperform Brazilian benchmarks.

2 Previous Research

Various recent researches have broadened the importance of regime switching models in different areas of finance. Hamilton, Harris, Hatzius, and West (2016) cited the paper of Garcia and Perron (1996) to debate the long-term trends of the real interest rate and the instability over time. Guidolin, Orlov and Pedio (2017) studied the effects of conventional monetary expansion and quantitative easing under a three-state Markov switching model. Giampietro, Guidolin and Pedio (2018) reported that regime switching models outperform single state models when applied to commodity pricing. Bensoussan, Hoe, Yan and Yin (2017) combined real options and game theory in different economic settings to demonstrate the regime switching in optimal investment decisions for market entry, such as the competing movements of leaders and followers.

To the best of our knowledge, prior to Campani and Garcia (2018), two papers aimed at the explicit solution of the investor problem using regime switching models in financial markets, that is, to maximize the utility from terminal wealth and/or future consumption by optimizing the asset allocation. Under finite horizon, Yin and Zhou (2003) developed a continuous-time Markowitz mean-variance portfolio in a problem with no consumption. Under infinite horizon, Sotomayor and Cadenillas (2009) maximized the expected utility from consumption to find the optimal portfolio and consumption policies to the log-utility investor as well as for the power utility investor. The infinite horizon has limited the latter research to capture and analyze horizon effects, and both researches were limited by working under fully observable regimes.

Under unobservable regimes and finite horizon, Honda (2003) found an analytical solution to the consumption and portfolio problems, but only for the very specific (and unrealistic) case where the investor would have a constant relative risk aversion equal to 0.5. For all other values, he used the martingale approach to numerically solve the optimal policies with Monte Carlo simulation. He worked with power utility from consumption and terminal wealth in a continuous time model with two hidden regimes, taking into account the expected mean return of the risky asset but not its volatility. All other parameters were constant and independent of the regime.

Bae, Kim and Mulvey (2014) studied the dynamic asset allocation of stock, bond and commodity markets, converting the static mean-variance model into an optimization problem under four unobservable regimes. They concluded that the information about the regimes can reduce the risk during left-tail events, such as crash periods. Jiang, Liu and Tse (2015) developed a dynamic investment strategy by applying regime switching models in asset allocation with iShares ETFs covering America, Europe and Asia to convey more information about the global systematic risk. Using two strategies, mean-variance and equally weighted portfolios, the authors found that a two-observable regime switching model outperforms the setting of a single regime in global portfolios. Zhang, Chen and Yao (2017) worked only with risky assets, without any risk-free asset, to demonstrate how the efficient frontier is affected by the shift in the market regime and investor's time horizon. Notwithstanding the importance of exploring the implementation of regime switching models in asset allocation, the limitation of these papers is the study under the Markowitz framework, *i.e.*, with a quadratic mean-variance utility function.

In Brazil, after Oliveira and Pereira (2014), Oliveira and Pereira (2018) studied asset allocation under two unobservable regimes (named as high and low markets) using two asset classes. Money was represented by the CDI rate as the riskless rate, and the stock market was represented by a selection of stocks that appear in all Ibovespa compositions during the observation period (January 2009 to December 2016). For most stocks, the low market denoted lower mean returns and lower volatility, while the high market represented higher mean returns and higher volatility. The authors applied mean and variance resulting from the regime switching models in two strategies: tangent and global mean-variance (MVP) portfolios. Each strategy was assembled using three models: single-regime economy, regime switching economy and inferring the next period regime. On top, only for the MVP, they also varied the restrictions for allowing (or not) short sales. In total, the authors have compared nine models and concluded that the tangent portfolio (less risk aversion) improved with the use of the regimes in periods of high in the financial market, while the MVP (more risk aversion) was little affected by the regime switching model. Our results will show that a greater number of regimes could better accommodate short-term movements within long-term trends for the high and low markets, and that money and stocks lack all the characteristics to capture the main states of the Brazilian economy. Government treasuries represent a very important asset class in Brazil, if not the most important. In some regimes, the excess returns of treasuries over the CDI rate have non-negligible volatilities. Moreover, not working with market indexes could potentially generate a selection bias. For example, their stocks selection reduced the class to only 24 companies, while there were 385 companies listed on the Brazilian stock exchange in 2009. To measure portfolio performance, the authors compared: accumulated returns, Sharpe index and variance. But the certainty equivalent loss enables a more comprehensive evaluation of the strategies: employing a Monte Carlo simulation, the certainty equivalent loss compares the model solution (suboptimal portfolio) to the simulated solution (optimal portfolio). Also, the strategies subjected to the mean-variance framework limited them to achieve a realistic solution to the investor problem.

Michaud and Michaud (2008) show that the quadratic optimization within the mean-variance analysis is empirically far from realistic since it uses only mean and variance of returns to represent the investor's utility and investment objectives, for example, it does not account for higher statistical moments. This is a critical limitation, since investors do not seem to behave this way. Also, the mean-variance analysis is a static single-period framework, not representing the dynamic model required for long-term investments, which means, for example, that hedging demands are simply ignored. Moreover, this analysis does not consider utility from consumption.

Under realistic frameworks in regime switching economies, exact analytical solutions have not yet been discovered, opening the way for approximate analytical solutions as opposed to numerical solutions. Analytical solutions allow us to evaluate optimal strategies deeper as, for example, the impact of market parameters are more easily revealed. The stochastic differential utility function introduced by Duffie and Epstein (1992) is empirically more relevant than all utility the functions used in previous frameworks since it disentangles investors risk attitudes over time and economy states (in fact, the power and the log utilities are nested in the stochastic differential utility function). Campani and Garcia (2018) innovated by applying this utility function to asset allocation under regime switching models. To solve the problem, they introduced an approximate solution that provides fairly accurate strategies when comparing the certainty equivalent losses of the approximation to the optimal solution obtained through the simulation. Additionally, this approximation overcomes the simulation burden limitation. The CG model enables the dynamic investors, under regime switching models, to make decisions in real-time.

We pioneer when applying the CG Model in the Brazilian financial market. For the first time, four unobservable regimes in a finite horizon setting were estimated to explore dynamic portfolio strategies in Brazil, and we show that these strategies have outperformed the benchmarks.

The rest of the dissertation is structured as follows. Section 3 describes the methodology to set the model and the accuracy assessment. The composition of the asset classes is detailed on

Section 4. The four-regime model and the solution for the optimal portfolios in various scenarios are presented in Section 5, while Section 6 concludes. The appendices detail the processes within the CG model to set the economy and the investor, and to find the portfolio weights.

3 Methodology

We generated the regime switching model according to Hamilton (1989) and the allocation according to the CG Model using a system of MatLab™ programs, which operates as following: first, we find the solution for the single regime economy, later used to contrast with the multivariate regime solution. Second, using the maximum likelihood, we estimate the model parameters: mean returns and volatilities of the assets, correlation matrices, and transition probabilities. We also estimate the filtered and smoothed probabilities, to then obtain the regime switching model. Third, with that, we find the allocation strategy through the CG model (approximate solution). Then we perform a Monte Carlo simulation to find the optimal solution to assess the accuracy of the approximate solution. Finally, we propose a real-time portfolio strategy based on the CG model, and compare the obtained results against the benchmarks of the market.

We set a continuous-time model in a frictionless and arbitrage-free financial market, incorporating $n+1$ different asset classes: the short-term riskless asset and n different risky assets. We applied the CG model with risk premiums to absorb the volatility of the riskless returns in the risky assets returns. We calculated the excess returns from t to $t+1$ as:

$$\frac{S_{n,t+1} - S_{n,t}}{S_{n,t}} = \frac{1 + ra_{n,t}}{1 + rf_t} - 1, \tag{1}$$

where, $S_{n,t}$ is the asset risk premium, $ra_{n,t}$ is the asset absolute return and rf_t is the riskless return, all in vector form. Thereby, the risk premium of the riskless asset is constant and equal to zero. Appendix A details the remaining processes of the economy following the CG model.

The utility function used by the CG model disentangles the relative risk aversion (γ) from the consumption (ψ) and investment decisions. However, we considered the elasticity of intertemporal substitution $\psi = \infty$, which was equivalent to limit the investor problem only to the assets allocation in the portfolio. In other words, this application does not predict consumption values. In funds, investors could take the consumption decisions individually. Following Campani and Garcia (2018) and Guidolin and Timmermann (2007), we considered the parameter that controls investor's attitudes over the states of nature (*i.e.*, states of the economy) as $\gamma = 5$, and time preference rate was set at a yearly rate of $\beta = 2\%$. Appendix B presents the investor processes according to the CG model, and appendix C shows the CG model computation method for the portfolio weights.

The model was built with premisses to generate an optimal unconstrained portfolio, *i.e.*, allowing short sales. The details used to estimate the Markov chain parameters are in appendix D. The estimation of matrix σ_π, a parameter introduced by the CG model, is detailed in appendix E. We adapted the model by discretizing the decision process, and used a discrete-time strategy in which the investor decides based on current information and waits until the next period to remake decisions with the new information acquired. The smaller the rebalancing period, the more similar it is to the continuous model. We defined it as one week, but did not include transaction costs in order to compare the results obtained with the market indexes.

The wealth equivalent loss was used to compare the solutions obtained through the CG Model, which are suboptimal since they originate from an approximation, with the optimal solutions obtained through the simulation. To measure it, a structure with a Monte Carlo simulation was formed combining investment horizons and two grids.[1] The horizons were a set from 13 to 260 weeks. The first grid covered all possible portfolio weights in 12.5% steps to allocate on the 3 risky assets and the riskless asset. The second grid consisted of all probabilities of being on each of the four regimes, also in 12.5% steps. We simulated $N = 30.000$ times on each combination node.[2] This exponential combination required 5 rounds of 6 virtual machines in parallel processing to complete the simulation.[3] Each round lasted approximately 127 hours.

The wealth equivalent loss was calculated as follows: we consider two identical investors, except that they invest different initial amounts. The first follows the optimal strategy, while the second follows the approximate strategy (suboptimal). The second investor starts with a wealth equal to $100. Given the two strategies, we match their value functions (utility functions) to calculate the initial wealth the first investor needs to have to achieve the same terminal wealth as second investor. As the first investor follows an optimal strategy, her initial wealth will be less than $100. The difference of between the investors' initial wealth is the percentage wealth equivalent utility loss due to the suboptimal strategy during the whole period (our research presents this percentage adjusted for the year base). If the loss is negligible, it means the model

[1] We followed the details of Campani and Garcia (2018) to build up the simulation process.
[2] Guidolin and Timmermann (2007) say that $N \geq 20,000$ times guarantees sufficient accuracy.
[3] The virtual machines ran in 2 computers with the following configuration: (i) 2.8 GHz Intel Core i7 processor, with 8GB 1333 MHz DDR3 memory; and (ii) 1.4 GHz Intel Core i5 processor, with 4GB 1600 MHz DDR3 memory.

provides fairly accurate and realistic strategies, and the investor can comfortably rely on it.

4 Brazilian Data

The model was composed by one riskless asset and three classes of risky assets, which were respectively chosen to capture the markets: money, fixed income, domestic stocks and international stocks (in Brazilian currency terms). These four classes were determined for being the most liquid options in the Brazilian financial market and because their accessibility for ordinary investors, who are less equipped to make complex assessments and use less sophisticated techniques than the institutional investors. We followed Andrino and Leal (2018) definition of the most liquid financial asset classes accessible to ordinary investors through the instruments available in the Brazilian market. The only exception was the authors organized inflation as one asset class to capture securities that reflect the cost of living adjusted performance, instead we named it as fixed income to comprehend more diversified treasuries into the session associated to the interest rate market. We followed their terminology for the remaining asset classes.

All asset classes were represented by a correspondent index (or rate) to mitigate selection bias. The time series were extracted from Economatica® database using a window of weekly observations from December 7th, 2001 to August 31st, 2018 – encompassing 873 observations.

4.1 Money Market

The Interfinancial Certificates of Deposit (CDI) rate represents an investment instrument widely used in Brazil and often used as the riskless rate for the Brazilian economy, as in the research of Andrino and Leal (2018) and Oliveira and Pereira (2018). It is computed by Cetip, which merged with BM&F Bovespa, the Brazilian Securities, Commodities and Futures Exchange, creating the company named B3 – Brasil, Bolsa, Balcão. Not only its computation is made by the private sector independently from the government institutions, but also the use of the CDI rate as a benchmark for the Brazilian risk-free rate is justified because it closely follows, in value and variation, the short-term treasury securities rates (Selic). Table 1 shows that the CDI annualized mean return and volatility respectively are 12.90% and 0.56%, while the same parameters of post-fixed interest-based treasuries linked to Selic (LFT) are 12.96% and 0.56% – and that they correlate at 99.99%. Essentially, the CDI rate can be considered a reliable proxy for the Brazilian basic interest rates, due to its strong correspondence to the most liquid assets. It was used to represent this asset class, named as money in the model.

4.2 Fixed Income Market

Brazil stands out for its high level of interest rates, and its fixed income market is so inviting that it discourages most investments in other local markets. Within this asset class, government treasury bonds are the most liquid securities. Anbima provides a broad spectrum of treasuries indexes.[4] Some of its widely used indexes are IMA-B for the inflation-based treasuries, such as NTN-B; and IRF-M for the pre-fixed interests-based treasuries, such as LTN. Among other types of indexes, Anbima offers IMA-G which represents a weighted portfolio of treasuries.

We worked with IMA-G to represent this asset class as it replicates the returns that diversified ordinary investors could obtain. Table 1 shows the performances of these indexes. Furthermore, IMA-G is the oldest Anbima index of treasury portfolios (still in operation), and yet, that was the limitation of our research to extend the window of observations. Therefore, IMA-G was selected to represent this asset class, denominated in the model as fixed income.

4.3 Domestic Stock Market

Ibovespa or IBrX 100 could be used to represent the domestic stock market, as most instruments for ordinary equity investors are linked to one of these B3's indexes. But we believe that the calculation method of IBrX 100 captures the market better. Ibovespa was previously based on liquidity, which allowed a stock in free falling prices, due to a greater trading volume, had its share actually increased in the index. Only in 2014, among other adjustments, this weighting method was changed to market capitalization with a liquidity cap. On the other hand, IBrX 100 is based in market capitalization since inception. In addition, IBrX 100 is comprised of the 100 largest stocks of the market, while Ibovespa does not have a predetermined number of stocks (Ibovespa consisted of 65 stocks in October 5$^{\text{th}}$, 2018). As a result of these distinctions, as shown in Table 1, IBrX 100 outperforms Ibovespa in terms of annualized returns and risks.

We did not segment domestic stocks into large and small caps (MCLX and SMLL indexes) due to the lack of liquidity and because the calculation of these B3 indexes only started in 2008. The lack of liquidity and a reduced time series resulted in unusual similar performances in relation to IBrX 100, as can be seen in Table 1. Consequently, this research utilized IBrX 100 to fully represent the domestic stock market asset class, referred to in the model as BR stocks.

[4] Anbima is the Brazilian Financial and Capital Market Entities Association.

4.4 International Stock Market

Considering the US Dollar exchange rate against the Brazilian Real is inversely related to the domestic stocks indexes, we used it as an alternative investment to reduce investor's specific (country) risks.[5] Additionally, when diversifying into US Dollars, Brazilian investors can use the available capital in hard currency to take advantage of international investment opportunities. From the Brazilian point of view, the index that best reproduces these opportunities would be the S&P 500. Although this is not the only proxy for international stock markets, it might be the most relevant one for Brazilians, as investing in other countries is not so common. Thus, the international stock market in Brazilian currency terms will be referred to as US stocks.

Instead of directly trading abroad, there are two simpler investment vehicles for Brazilian investors to access the US stocks: exchange-traded funds (ETF) and domestic mutual funds that concentrate the majority of investments abroad. Unfortunately, ETFs are restricted to qualified investors. On the other hand, mutual funds are available to ordinary investors on well-known platforms. To capture both movements, the US stocks were defined as the compounded returns of S&P 500 in Dollars, along with the returns of USD/BRL rate (given by the PTAX rate). Considering our window of observations, annualized mean returns and volatilities, respectively were: S&P 500 7.10% and 16.61%; PTAX 4.55% and 15.66%; US stocks 10.66% and 16.93%.

4.5 Other Markets

We have verified other potential asset classes, which we did not consider in the application of the model: corporate bonds, mutual funds and real estate. First, we studied corporate bonds with two Anbima indexes: IDA DI (bonds linked to the CDI rate) and IDA Geral (bonds linked to the CDI or inflation rates), but we concluded that the underlying assets of these indexes present low liquidity. Thus, corporate bonds were not considered a class for ordinary investors to diversify in Brazil. Second, we investigated mutual funds as an asset class. Andrino and Leal (2018) explained that instead of the type of US hedge funds; in Brazil, the Anbima hedge funds index (IHFA) is used to benchmark the multi-market mutual funds. We did not consider multi-market mutual funds as a specific asset class, since the diversification that they would add to the model could be accessed by combining the already existing asset classes. Third, although real

[5] Meurer (2005) showed a negative effect between external capital flows and the Brazilian stocks performance. Groppo (2006) concluded that the major sensitivity of Brazilian stocks occurs due to the exchange rate. Ferreira and Zachis (2012) detected a negative correlation between the Brazilian stock indexes and the exchange rate. While, Naresh, Vasudevan, Mahalakshmi and Thiyagarajan (2017) indicated that the appreciation in the value of BRICS currencies against US Dollars had increased the value of the stock indexes of the correspondent countries.

estate could be considered a relevant asset class for diversification in Brazil, the B3 benchmark for the real estate funds (IFIX) only began to be computed in December 2010. We tested the model with IFIX, but the reduced window of observations subjected the estimation of the regimes to excessive short term influences. The other real estate index (IGMI-C) dates back to 2000, but it is only quarterly computed by IBRE/FGV/RJ, and does not include residential real estate. Therefore, we also did not include real estate as an asset class in the model.

5 Results

We identified the presence of $m = 4$ regimes in Brazil following Guidolin and Timmermann (2007).[6] The characteristics of our second regime were more similar to a bear market than to a slow growth environment, as originally defined by the authors. So we adopted a different terminology for the second regime. In the fourth regime, the literature named it referring to an optimistic recovery coming from a crash situation, but in Brazil it is not the most probable transition. However, we kept the fourth and the remaining names unchanged for simplicity. We highlight that our results refer to the Brazilian economy states, including the US stocks (defined in local currency terms). In ascending order, the regimes are: crash, bear, bull and recovery.

5.1 Model Specification

Table 2 presents the parameters estimated for the CG Model using the filtered probabilities (only with the information available before time t). The crash regime is characterized by the contrast of a strong negative risk premium of the BR stocks against the highest risk premium of fixed income, while the US stocks are only moderately negative. During this regime, all volatilities are high. The bear market is characterized by low negative risk premiums of the BR stocks and fixed income (in fact, the latter is practically zero), and a strong positive risk premium of the US stocks. The bear market has relatively low volatilities on both equity classes, but very high volatility in the fixed income. In the bull market, the BR stocks finally turn into positive territory. The fixed income risk premium is also positive, at the same time that the US stocks become negative again. Here, all volatilities are low. The recovery is the most optimistic scenario for the BR stocks, while the risk premiums of the competing assets are negative. Again, fixed income risk premium is only slightly negative, but here it also has low volatility. During

[6] Guidolin and Timmermann (2007) show that a model with fewer states is clearly misspecified.

the recovery regime, the US stocks have their strongest negative performance (in Reais).

Correlations show similar trends through the regimes. Except for a change of pattern in the bear market, correlations are stronger during the crash and are less intense when states become more optimistic. For example, the correlation between the BR stocks and fixed income can be as high as 46.6% during the crash regime, and as low as 19.0% during the recovery regime.

The transition probabilities show that the crash regime is almost certainly followed by a bull market, with an average duration of 9 weeks. The bear market is the most persistent regime, lasting on average 40 weeks. It is likely to be followed by bull or crash regimes. The bull market, another very persistent regime, lasts about 33 weeks and it is likely to be followed by the bear market. The recovery, the most optimistic state, can be followed by any other regime (unfortunately crash is the most likely). Table 3 presents the ergodic probabilities: over the long term (steady state) crash and recovery represent only 7.0% of the regimes, while bear and bull markets, the most common regimes, respectively appear in 41.6% and 44.4% of all periods.

5.2 Model's Benchmark

Table 4 helps to understand the intuition behind the estimated parameters since it presents the absolute returns of the assets. As this is a historical analysis (not a strategy), we were able to use the smoothed probabilities to segment the returns. Starting with money, the lowest returns occurred in the bear market, while the highest returns were during the recovery regime. In turn, fixed income presented a different behavior. During the crash, this class performed best in comparison to its own results during other regimes and also against the other market segments, except for the PTAX. But during the bear and recovery regimes, the fixed income performed very close to money. The BR stocks were in an upward trend of returns from regimes 1 to 4 (crash to recovery), which can be very intuitive. In contrast, the subdivision of the US stocks class demonstrates that the PTAX had a strong downward trend of returns form regime 1 to 4, which explains the returns of the class formed by its combination with the S&P 500.

To analyse the Transition Probabilities, first, it is reasonable to compare the results we obtained against the benchmark provided by Campani and Garcia (2018). Although we worked with a different window of observations, the comparison is valid to see that regimes 1, 2 and 4 had similar durations in Brazil and in the US. The main difference was in regime 3 (bull market), which lasted much longer in the US. Another contrast were the probabilities of the most extreme regimes. In the US, a crash was almost surely followed by a recovery; in Brazil, it was almost surely followed by the bull market. On the other hand, the recovery regime had 8.4% chance of being followed by a crash in the US, while in Brazil that chance was only 2.00%.

Secondly, according to the filtered probabilities, figure 1 shows that the recovery regime was more present only until 2006. The crash and bull regimes were more frequent until 2012. After that, the bear market became the most persistent regime, lasting up to $t + 1$ after the end of the window of observations. Figures 2 and 3, respectively reflect the dynamics of the filtered and the smoothed probabilities. It can be said that the first represents the view of the investor and the second, the view of the analyst. The investor makes decisions for the next exercise, working with less information (filtered). While the analyst studies past events, accessing the entire time series. Both show that the model was capable of capturing important economic events, such as the presidential impeachment of 2016, reflected by the crash regime. Figure 4 shows the accumulated risk premiums at each observation point and the most likely regime according to the smoothed probabilities, to highlight, for example, the volatility related to the 2002 elections, the 2008-2009 subprime crisis, and the 2017-2018 positive spikes in the Brazilian stock market. We believe that the analysis of those different perspectives provides economic evidence to accept the results obtained for the transition probabilities between the regimes.

5.3 Dynamic Strategy

The dynamic strategy was set with horizons of 13 to 260 weeks, differently from Guidolin and Timmermann (2007) and Campani and Garcia (2018). Accessing monthly observations (over 45 years), they could study horizons up to 10 years. In Brazil, the selected data allowed us to go only up to 17 years ago. As a result, we opted for weekly data to increase observations. But the weekly data increased the simulation burden in the ratio of 120 to 260 in relation to the literature. Then, we shortened the vector of horizons to keep the simulation viable.

Table 5 reports the dynamic strategies and the wealth equivalent loss, comparing the solutions obtained through the model with those of the simulation. The comparison is illustrated by two scenarios. First, the known next period regime represents the filtered probabilities indicating 100% chance of occurring one specific regime (for each of the four regimes). Second, the unknown next period regime denotes the case where the investor admits being in a multi-regime economy, but does not have access to the filtered probabilities to infer the regimes, and then considers the most conservative configuration, which are the ergodic (long-run) probabilities.

During the crash regime, a 5-year horizon investor short sells both equity classes but mainly money, to invest in fixed income (1280%). The preference for heavily short selling money rather than BR stocks is due to the volatility of the later class being at its peak. In the bear market, overall leverage is reduced. Fixed income (339%) and US stocks (198%) are the main components of the portfolio, while investor remains sold on BR stocks and money. The preference for fixed income instead of money occurs despite the slightly negative risk premium. This is due to fixed income having a strong negative correlation with the US stocks, acting as an insurance for the investment in the asset class with the most positive returns during the bear market. In the bull market, the leverage of fixed income is again increased (2582%), and investor also buys BR stocks (42%), short selling the other classes. Recovery, as an optimistic state, indicates the greatest leverage. The investor short sells fixed income and US stocks, and loads up on money (4589%) and BR stocks (135%). Fixed income is slightly negative as in bear market, but here its correlations do not act as an insurance, thus the model indicates strongly short selling it.

Uncertainty significantly affected the portfolio allocation, because it has limited the investor to track the market, such as it could be done with the filtered probabilities. If the information about the next period regime is unclear, the strategies can not be so aggressive. In the unknown next period regime, allocation is virtually restricted to two long-and-short strategies: one between the interest-rate based assets and the other between the equity classes. This highlights the importance of using the regime switching model for an optimal allocation in Brazil.

All horizons of known next period regime showed negligible losses of less than 0.4% per year. When the next period regime is unknown, the losses increased progressively with the horizons. However, the losses of the unknown next period regime can also be considered negligible due to the frequency distribution of the filtered probabilities. Figure 5 shows the filtered probabilities separated by their size in relation to the other probabilities at each observation point. For example, the first largest probability denotes the most likely regime to occur, simultaneously, the smallest probability represents the less likely regime (this classification combines all regimes). The mean of the first largest probabilities was higher than 86%, while its minimum never fell below 36%. For example, equal probabilities (25% chance for each regime) did not occur in our sample. The filtered probabilities rarely presented the most likely regime with small percentages. Empirically, the most frequent scenario was one very high probability, while the other probabilities remained low. This scenario is represented by the known next period regime.

Figure 6 plots the portfolio weights of a 5-year horizon investor, who used the filtered probabilities to determine the allocation. On average, fixed income represents the largest share of the portfolio, while money leverages this portion. These positions alternate over the regimes. For the share in equity markets, the investor usually chooses BR or US stocks, but not both. We observe that the leverage in asset classes related to interest rates (fixed income and money) was much higher than the leverage of the two equity classes. We highlight that it does not refer to a direct short sales of fixed income or money securities, for example. Instead, we assumed that it is possible for our investor to trade an equivalent position in the derivatives market.

5.4 Dynamic Strategy's Benchmark

Table 6 shows the results obtained through the CG model for 52 weeks, and the benchmarks in terms of risk premium and volatilities. Since inception until 2012, the crash, bull and recovery regimes were more frequent. Then, we observe the highest risk premiums of fixed income were slightly above 3% in 2003 and 2009, and that the highest risk premiums of the BR stocks were above 30% in 2003, 2007 and 2009. The US stocks were negative from 2002 to 2010. At the same time, in 6 of these 10 years of more leveraged regimes, through the CG model, the investor could have obtained risk premiums above 50% per year − being 2003, 2005 and 2006 above 95%. Only in 2007 and during the subprime crisis, the model did not outperform the market. Observing the volatilities, disregarding 2008, fixed income remained below 0.3% and the equity classes below 4.3%. In the same period, the volatility of the model did not exceed 4.6%.

After 2012, the bear market became the most persistent regime in Brazil. We can observe that the risk premiums of fixed income and BR stocks presented strong oscillations, while US stocks (in Reais) presented consistent positive double-digits risk premiums (except for 2016). As the strategy during this regime uses less leverage, the results produced by the model changed in magnitude, yet they have outperformed the market in 4 of these 6 years. Except 2017, the model produced risk premiums above 33% per year. In 2017, after an exceptionally long bear market, the bull regime *tried* to be present again. The only relevant negative result produced by the model originated from this situation, unique in our sample, but which requires attention.

The absolute returns produced by the model are shown in the last column of table 6. Considering the same moving windows of 52 weeks, we observe that the highest absolute return of the CG model was above 247% in March 2007, while the lowest performance was -54.7% in May 2017. But in particular, what attracted the most attention were the average absolute returns produced by the model: since inception, they have exceeded 71% per year. When we consider a more recent interval, since 2012, it has produced absolute returns above 43% per year.

6 Conclusion

This dissertation applied the approximate analytical solution of Campani and Garcia (2018) to the portfolio problem with the stochastic differential utility function from Duffie and Epstein (1992) under a four regime economy. Our research shows that the solution is fairly accurate in the Brazilian financial market with the asset classes: money, bonds, BR stocks and US stocks.

We could observe that the filtered probabilities have a significant impact on the portfolio weights, since the assets performances strongly depend on the economy states. Moreover, under this realistic framework, the use of simulations to obtain optimal solutions for dynamic portfolios, for practical reasons, is extremely limited by the simulation burden. Instead, we applied the approximation given by the CG model, which enabled a real-time solution.

The strategy obtained through the CG model outperformed the Brazilian benchmarks: CDI, IMA-G, IBrX 100 and S&P 500 (in local currency terms). Although this assessment was limited by not considering transaction costs and by an in-sample estimation of the model parameters, the results of this dissertation present interesting potential and call attention to a deeper analysis.

Our research has pioneered when applying regime switching models to explore, in real-time, dynamic portfolio strategies in Brazil. The analysis conducted here provides evidence to accept that four unobservable regimes exist in the asset returns of this economy. More importantly, we showed that working with regimes is key to achieve good performance with portfolio strategies.

Future researches? We see complementary development to be done in this field of study. If a reliable predictor is inaugurated for the Brazilian economy with the same periodicity and window of observations, it could be tracked to improve the model. The performance of the strategy suggested by this article could be further investigated in light of transaction costs and out-of-sample model estimation. However, the out-of-sample estimation will depend on the number of observations available, since our suggestion would be to use moving windows from real data observations to estimate the model for the following period (at each time t). Also, a further investigation of the preference parameters (γ, β and ψ) to better match institutional investors could improve the performance. Another idea would be to include more asset classes, such as real estate and private equity funds, for example. Last, but not least, we would be very interested in analyzing asset-liability management strategies under a regime switching economy.

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Appendices

A Setting the Economy

Following the CG model, the dynamics of the risk premium ($\frac{dS_t}{S_t}$) of the n risky assets are:

$$\underbrace{\begin{bmatrix} \frac{dS_{1,t}}{S_{1,t}} \\ \frac{dS_{2,t}}{S_{2,t}} \\ \dots \\ \frac{dS_{n,t}}{S_{n,t}} \end{bmatrix}}_{\frac{\mathbf{dS_t}}{\mathbf{S_t}}} = \underbrace{\begin{bmatrix} \mu_{1,t} \\ \mu_{2,t} \\ \dots \\ \mu_{n,t} \end{bmatrix}}_{\mu_{\pi,t}} dt + \underbrace{\begin{bmatrix} \sigma_{11,t} & 0 & \dots & 0 \\ \sigma_{21,t} & \sigma_{22,t} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \sigma_{n1,t} & \sigma_{n2,t} & \dots & \sigma_{nn,t} \end{bmatrix}}_{\sigma_{s,t}} \underbrace{\begin{bmatrix} \Delta Z_{1,t} \\ \Delta Z_{2,t} \\ \dots \\ \Delta Z_{n,t} \end{bmatrix}}_{\mathbf{dZ_t}}, \tag{2}$$

where, $\mu_{s,t}$ represents the column vector with instantaneous expected risk premium; and $\sigma_{s,t}$ is the volatility matrix designed to be lower triangular with absolutely no loss of generality; and $Z_{1,t}$, $Z_{2,t}$, ..., $Z_{n,t}$ are standard and independent Brownian motion processes.

The risky assets drift vector ($\mu_{s,t}$) and the volatility matrix ($\sigma_{s,t}$) declared on equation (2) are time-varying and function of an unobservable state variable Y_t. The state variable Y_t is an independent continuous-time Markov chain, right-continuous and admitting only values in $R = 1, 2, ..., m$. While, R represents the finite set of m possible regimes in the economy.

On state $Y_t = i, i \in R$, we have:

$$\mu_{s,t} = \begin{bmatrix} a_{1,i} \\ a_{2,i} \\ \dots \\ a_{n,i} \end{bmatrix} \quad \text{and} \quad \sigma_{s,t} = \sigma_{s,i} = \begin{bmatrix} \sigma_{11,i} & 0 & \dots & 0 \\ \sigma_{21,i} & \sigma_{22,i} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ \sigma_{n1,i} & \sigma_{n2,i} & \dots & \sigma_{nn,i} \end{bmatrix}, \tag{3}$$

with coefficient $a_{j,i}$ and matrices σ_i constant for every $i \in R$ and $j = 1, 2, ..., n$. The risky assets drifts are regime dependent and simultaneously time-varying even if the regime remain unchanged. They are collected on the time-varying drift matrix $\mathbf{D_{s,t}}$ with dimensions nxm:

$$\mathbf{D_{s,t}} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,m} \\ a_{2,1} & a_{2,2} & \dots & a_{2,m} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,m} \end{bmatrix}. \tag{4}$$

Advancing with the model onto the economy states, we considered an economy ruled by a regime-switching unobservable state variable Y_t, with the characteristics previously presented. The regime switching process Y_t, starting at any random time t_0 on any given state, remains there for an exponentially distributed length of time, and then jumps to another state.[7]

The behavior of staying at one state then jumping to another is treated through the transition probabilities. More precisely, given that the current state is i, the probability of jumping to another state j over the next Δt time period is $P_{ij,\Delta t} = \frac{\lambda_{ij}}{\sum\limits_{k\neq i} \lambda_{ik}} \left(1 - e^{-\sum\limits_{k\neq i} \lambda_{ik}\Delta t}\right)$, with $j \neq i \in R$ and $\lambda_{ij} \geq 0$. We define $\lambda_{ii} = -\sum\limits_{k\neq i} \lambda_{ik} \leq 0$ such that $P_{ij,\Delta t} = \frac{\lambda_{ij}}{-\lambda_{ii}} \left(1 - e^{\lambda_{ii}\Delta t}\right)$. Therefore, the probability of *staying* in the same regime i over the next Δt time period is $P_{ii,\Delta t} = e^{\lambda_{ii}\Delta t}$.

The parameters λ_{ij} $(j \neq i \in R)$ are assumed constant and represent the density of transition probabilities from regime i to regime j. The closer to zero λ_{ii} is, the more persistent is regime i. When $\lambda_{ii} = 0$, once the economy jumps to state i, it remains there onwards. So, the CG Model could be understood as a generalization of the standard single regime model, since this particular case is nested, *i.e.*, if we assume $\lambda_{ii} = 0$ and that the initial state of the economy is already state i. Another standard intrinsic assumption of the CG Model is that the inter-regime times are independent, and they are also independent of all risky assets Brownian motions.

B Setting the Investor

In the CG model, the investor is considered *small*, in the sense that her attitudes do not affect market prices, being only a price taker. Labor and other types of income are disregarded. As the states are unobservable, we assume the investor sees the following multi-asset process:

$$\frac{\mathbf{dS_t}}{\mathbf{S_t}} = \mathbf{D_{s,t}}\pi_\mathbf{t}\,dt + (\mathbf{V}\pi_\mathbf{t})\,\mathbf{dZ_t}, \tag{5}$$

where, $\frac{\mathbf{dS_t}}{\mathbf{S_t}}$ is the column vector with all n risky assets processes $(\frac{dS_{j,t}}{S_{j,t}})$, $\mathbf{D_{s,t}}$ is the element set by equation (4), and $\pi_\mathbf{t}$ is an m x 1 column vector storing the probabilities of being in each possible economy state at time t, conditioned on the available information at the same time t:

$$\pi_\mathbf{t} = \begin{bmatrix} \pi_{1,t} & \pi_{2,t} & \dots & \pi_{m,t} \end{bmatrix}^T, \tag{6}$$

[7] A property of the exponential distribution is that it is memoryless, in accordance to the Markov property. This *memorylessness* can be stated as: $P(T > s + t|T > s) = P(T > t)$ for all $s, t \geq 0$, where T is the final horizon.

the matrices $\sigma_{\mathbf{s,i}}$, which are constant and specific to the regime ($i \in R$), are detailed below:[8]

$$\sigma_{\mathbf{s,i}} = \begin{bmatrix} \sigma_{11,i} & 0 & ... & 0 \\ \sigma_{21,i} & \sigma_{22,i} & ... & 0 \\ ... & ... & ... & 0 \\ \sigma_{n1,i} & \sigma_{n2,i} & ... & \sigma_{nn,i} \end{bmatrix}, \tag{7}$$

and $\mathbf{V}$ is a $1 \ x \ m$ row vector of matrices such as below:[9]

$$\mathbf{V} = \begin{bmatrix} \sigma_{\mathbf{s,1}} & \sigma_{\mathbf{s,2}} & ... & \sigma_{\mathbf{s,m}} \end{bmatrix}. \tag{8}$$

The state probabilities in equation (6), given by $\pi_{\mathbf{t}}$, were treated as state variables in the CG Model. The behavior of the probabilities in $\pi_{\mathbf{t}}$ are only determined from the risky assets returns. Thence, we can assume they adopt the following process:

$$\underbrace{\begin{bmatrix} d\pi_{1,t} \\ d\pi_{2,t} \\ ... \\ d\pi_{m,t} \end{bmatrix}}_{d\pi_{\mathbf{t}}} = \underbrace{\begin{bmatrix} \sum_{i=1}^{m} \lambda_{i1}\pi_{i,t} \\ \sum_{i=1}^{m} \lambda_{i2}\pi_{i,t} \\ ... \\ \sum_{i=1}^{m} \lambda_{im}\pi_{i,t} \end{bmatrix}}_{\mu_{\pi,\mathbf{t}}} dt + \underbrace{\begin{bmatrix} \sigma_{11,\pi} & \sigma_{12,\pi} & ... & \sigma_{1n,\pi} \\ \sigma_{21,\pi} & \sigma_{22,\pi} & ... & \sigma_{2n,\pi} \\ ... & ... & ... & ... \\ \sigma_{m1,\pi} & \sigma_{m2,\pi} & ... & \sigma_{mn,\pi} \end{bmatrix}}_{\sigma_{\pi}} \mathbf{dZ_t^*}, \tag{9}$$

with the suitable definition $\sigma_{\mathbf{i},\pi} = \begin{bmatrix} \sigma_{i1,\pi} & \sigma_{i2,\pi} & ... & \sigma_{in,\pi} \end{bmatrix}$ as a row vector for all $i \in R$.[10] Here, $d\pi_t$ denotes the probabilities process, $\mu_{\pi,t}$ represents the probabilities of being on each regime, and σ_π is the probability of shifting between regimes (noting that $\sigma_{11,\pi}, \sigma_{22,\pi}, ..., \sigma_{nn,\pi}$ are the probabilities of staying at the same regime). The drift above was chosen in accordance with theorem 9.1 of Liptser and Shiryaev (2000).

The investor optimally chooses consumption C_t, defined as the rate of consumption at the instant t. She consumes $C_t dt$ over the time interval from t to $t + dt$, and invests a fraction $\alpha_{1,t}$ of her total wealth W_t in the risky asset 1, and so on till a fraction $\alpha_{n,t}$ in the risky asset n,

[8] The regime-dependent volatility matrices are drawn to be lower triangular with absolutely no loss of generality.
[9] Alternatively, Campani and Garcia (2018) tested a model in which the investor averages the variance-covariance matrices, instead of the volatility matrices. They state the results were virtually the same.
[10] In fact, one of the probability processes is redundant, given that all probabilities must sum up to one. However, we prefer to keep this redundancy for sake of clarity.

while the rest is put in the short-term riskless bond. Her wealth dynamics will thus write:

$$dW_t = W_t \boldsymbol{\alpha_t} \frac{\mathbf{dS_t}}{\mathbf{S_t}} - C_t dt = W_t \boldsymbol{\alpha_t} \left[\mathbf{D_{s,t}} \boldsymbol{\pi_t} dt + (\mathbf{V}\boldsymbol{\pi_t})\, \mathbf{dZ_t}\right] - C_t dt \tag{10}$$

where $\boldsymbol{\alpha_t} = [\alpha_{1,t}\ \alpha_{2,t}\ \ldots\ \alpha_{n,t}]$. If we considered that our investor's preferences were over consumption and terminal wealth (bequest), these preferences would be represented in the CG model by a continuous-time recursive utility function as follows:

$$J_t = E_t \left[\int_{u=t}^{T} f(C_u, J_u)\, du + \frac{W_T^{1-\gamma}}{1-\gamma} \right]. \tag{11}$$

Duffie and Epstein (1992) proved that this formulation, known as the stochastic differential utility function, is valid with bequest (time T is the investor's horizon). In equation (11), $f(C, J)$ is a normalized aggregator of the consumption rate, and the continuation utility takes the form:

$$f(C, J) = \frac{\beta}{1-\frac{1}{\psi}} (1-\gamma) J \left\{ \left[\frac{C}{[(1-\gamma) J]^{\frac{1}{1-\gamma}}} \right]^{1-\frac{1}{\psi}} - 1 \right\}. \tag{12}$$

Campani and Garcia (2018) demonstrated that predicting consumption has limited impact over the optimal portfolio. Thus, we limited the model disregarding consumption attitudes ($\psi = \infty$). By that, we reduced the stochastic differential utility function (equation 11) to the power utility function, as used by Guidolin and Timmermann (2007).

C Solving the Problem

At the CG model, the problem with no consumption is defined by $\psi = \infty$. Thus, all (power) utility comes from the terminal wealth. In this case, the Bellman equation could be simplified and the equation that will solve this recursive framework problem is:

$$0 = \sup_{\{\alpha_t\}} \left[\begin{array}{c} -\beta J_t + \frac{\partial J_t}{\partial t} + J_W W_t \boldsymbol{\alpha_t} \mathbf{D_{s,t}} \boldsymbol{\pi_t} + \frac{1}{2} J_{WW} W_t^2 \boldsymbol{\alpha_t} (\mathbf{V}\boldsymbol{\pi_t}) (\mathbf{V}\boldsymbol{\pi_t})^{\mathbf{T}} \boldsymbol{\alpha_t^{\mathbf{T}}} + \\ + \sum_{i=1}^{m} J_{\pi_i} \sum_{j=1}^{m} \lambda_{ji} \pi_{j,t} + \frac{1}{2} \sum_{i=1}^{m} J_{\pi_i \pi_i} \boldsymbol{\sigma_{i,\pi}} \boldsymbol{\sigma_{i,\pi}^{\mathbf{T}}} + \sum_{i=1}^{m} J_{W\pi_i} W_t \boldsymbol{\alpha_t} (\mathbf{V}\boldsymbol{\pi_t}) \overline{\boldsymbol{\sigma}}_{\mathbf{i},\pi}^{\mathbf{T}} + \sum_{i<j} J_{\pi_i \pi_j} \boldsymbol{\sigma_{i,\pi}} \boldsymbol{\sigma_{j,\pi}^{\mathbf{T}}} \end{array} \right], \tag{13}$$

where $\overline{\boldsymbol{\sigma}}_{\mathbf{i},\pi}^{\mathbf{T}}$ represents the transposed vector $\boldsymbol{\sigma_{i,\pi}}$ ($i \in R$) without the last element.[11] The subscripts denote partial derivatives (except t, which merely denotes the value at time t). The

[11] This is just an adjustment to make matrix dimensions agree. After all, these last terms refer to an independent Brownian motion with no correlation with the wealth dynamics.

function $V(W_t, \pi_t, t) = \sup_{\{\alpha_t\}}[J(W_t, \pi_t, t)]$ is the value function for this problem, and it depends on the observable state variables of the economy.

The first-order condition for the portfolio weight gives the following:

$$\alpha_t = \frac{V_w}{-V_{ww}W_t}(\mathbf{D_{s,t}}\pi_t)^{\mathbf{T}}\left[(\mathbf{V}\pi_t)(\mathbf{V}\pi_t)^{\mathbf{T}}\right]^{-1} + \sum_{i=1}^{m}\frac{V_{w\pi_i}}{-V_{ww}W_t}\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_t)^{-1}. \tag{14}$$

We now substitute the optimal expression for portfolio strategies, equation (14), into equation (13) to obtain the final Bellman equation for the problem under recursive utility:

$$\begin{aligned} 0 &= -\beta V_t + \frac{\partial V_t}{\partial t} + V_w W_t r + \\ &+ \sum_{i,j=1}^{m} V_{\pi_i}\lambda_{ji}\pi_{j,t} + \frac{1}{2}\sum_{i,j=1}^{m}\left(V_{\pi_i\pi_j}\sigma_{\mathbf{i},\pi}\sigma_{\mathbf{j},\pi}^{\mathbf{T}} - \frac{V_{w\pi_i}V_{w\pi_j}}{V_{ww}}\overline{\sigma}_{\mathbf{i},\pi}\overline{\sigma}_{\mathbf{j},\pi}^{\mathbf{T}}\right) - \\ &- \frac{1}{2}\frac{V_w^2}{V_{ww}}(\mathbf{D_{s,t}}\pi_t)^{\mathbf{T}}\left[(\mathbf{V}\pi_t)(\mathbf{V}\pi_t)^{\mathbf{T}}\right]^{-1}(\mathbf{D_{s,t}}\pi_t) - \\ &- \sum_{i=1}^{m}\frac{V_w V_{w\pi_i}}{V_{ww}}\overline{\sigma}_{\mathbf{i},\pi}(\mathbf{V}\pi_t)^{-1}(\mathbf{D_{s,t}}\pi_t). \end{aligned} \tag{15}$$

The problem admits the following wealth-separable solution:

$$V(W_t, \pi_t, \tau) = H(\pi_t, \tau)\frac{W_t^{1-\gamma}}{1-\gamma}, \tag{16}$$

in which $\tau = T - t$ is the time left to horizon, and terminal condition states that $H(\pi_t, 0) = 1$. The bad news is that it is virtually impossible to find the exact functional form of $H(\pi_t, \tau)$. The good news is that, utilizing the approximation introduced by Campani and Garcia (2018), it is possible to find an approximation for H. The authors proposed to consider $\mathbf{V}\pi_t$ as a constant term given by its long-run (ergodic) value:

$$(\mathbf{V}\pi_t) \approx (\mathbf{V}\pi_\infty). \tag{17}$$

They claim this approximation is fairly accurate to achieve an optimal portfolio with maximized utility. Thus, we can find the approximation for H as follows:

$$H(\pi_t, \tau) = \exp\left[A_0(\tau) + \sum_{i=1}^{m} A_i(\tau)\pi_{i,t} + \sum_{i=1}^{m} B_i(\tau)\pi_{i,t}^2 + \sum_{i<j} C_{ij}(\tau)\pi_{i,t}\pi_{j,t}\right]. \tag{18}$$

All coefficients above are time-varying and solve a system of ordinary differential equations with boundary conditions equal to zero when $\tau = 0$ (*i.e.*, at maturity). With this approximation, the optimal portfolio strategy is given by:

$$\begin{aligned} \alpha_{\mathbf{t}} &= \frac{1}{\gamma}\left(\mathbf{D}_{\mathbf{s},\mathbf{t}}\pi_{\mathbf{t}}\right)^{\mathbf{T}}\left[\left(\mathbf{V}\pi_{\mathbf{t}}\right)\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{\mathbf{T}}\right]^{-\mathbf{1}} + \\ &+ \frac{1}{\gamma}\sum_{i=1}^{m}\left[A_i(\tau) + 2B_i(\tau)\,\pi_{i,t} + C_{pi}(\tau)\,p_t + \sum_{j\neq i} C_{ij}(\tau)\,\pi_{j,t}\right]\overline{\sigma}_{\mathbf{i},\pi}\left(\mathbf{V}\pi_{\mathbf{t}}\right)^{-\mathbf{1}}. \end{aligned} \tag{19}$$

D Estimation of Parameters

The following parameters were estimated using the maximum likelihood according to Hamilton (1989). We assume that a weekly period is short enough such that we can reasonably match Hamilton's transition probabilities with our densities of transition probabilities:[12]

$$\text{Prob}\{Y_{t+1} = j | Y_t = i\} = P_{ij,\Delta t=1} = P_{ij} = \frac{\lambda_{ij}}{-\lambda_{ii}}\left(1 - e^{\lambda_{ii}}\right), \tag{20}$$

in which we have conveniently chosen the time unit as the same as the period length (that is, one week). We will then have the following identities:

$$\lambda_{ii} = \ln P_{ii}, \text{ and} \tag{21a}$$

$$\lambda_{ij} = -\frac{P_{ij}\ln P_{ii}}{1 - P_{ii}}. \tag{21b}$$

We collect the (constant) discrete-time probabilities in a matrix we call **P**:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix}. \tag{22}$$

The investor uses the filter explained in Hamilton (1989) to infer the current regime and take her optimal investment decisions. This optimal inference relies on iterating the equations below:

$$\widehat{\mathbf{Y}}_{\mathbf{t}|\mathbf{t}} = \frac{\widehat{\mathbf{Y}}_{\mathbf{t}|\mathbf{t-1}} \circ \eta_{\mathbf{t}}}{\mathbf{1}^{\mathbf{T}}\left(\widehat{\mathbf{Y}}_{\mathbf{t}|\mathbf{t-1}} \circ \eta_{\mathbf{t}}\right)}, \tag{23a}$$

[12] Note that this would be exact if the period considered were the infinitesimally short period dt.

$$\text{and} \quad \widehat{\mathbf{Y}}_{\mathbf{t+1|t}} = \mathbf{P^T}\widehat{\mathbf{Y}}_{\mathbf{t|t}}, \tag{23b}$$

where $\mathbf{1}$ represents here an $m\text{x}1$ vector of ones and the symbol $\circ$ denotes the element-by-element multiplication. $\widehat{\mathbf{Y}}_{\mathbf{t|t}}$ and $\widehat{\mathbf{Y}}_{\mathbf{t+1|t}}$ are also m x 1 vectors containing the updated probabilities of having each regime running respectively at times t and $t+1$, given the available information at time t. Finally, $\eta_\mathbf{t}$ is another m x 1 vector whose elements are the probability densities of the risky assets returns at time t, conditioned by being on each of the states. Equation (23a) uses the current information (*i.e.*, the risky assets returns) at time t to update the probabilities of each of the states in this last period t. Equation (23b) uses this update of the economy regime to optimally estimate the probabilities of being in each state in the next period (next week). In order to put in practice the iteration above, we need a starting point. Obviously, this can be naturally guessed by the investor, based on her beliefs of the initial state of the economy. In this study, we will start from the long-run regime probabilities (often called ergodic or unconditional probabilities), which are given by:

$$\widehat{\mathbf{Y}}_{\mathbf{1|0}} = \left(A^T A\right)^{-1} A^T e_{m+1}, \tag{24}$$

where e_{m+1} denotes the last column vector of the identity matrix of order $m+1$ and A is an $(m+1)$ x m matrix in which the first m rows are the rows of $I_m - P^T$ (I_m is the identity matrix of order m) and the last row has only 1's.[13] To find vector $\eta_\mathbf{t}$, we recall the process followed by the assets, if on a single-regime economy, admit the solutions:[14]

$$\begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \ldots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \end{bmatrix} = \begin{bmatrix} \mu_{1,t} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{1k,t}^2 \\ \mu_{2,t} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{2k,t}^2 \\ \ldots \\ \mu_{n,t} - \frac{1}{2}\sum\limits_{k=1}^{n} \sigma_{nk,t}^2 \end{bmatrix} + \begin{bmatrix} \sigma_{11,t} & 0 & \ldots & 0 \\ \sigma_{21,t} & \sigma_{22,t} & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ \sigma_{n1,t} & \sigma_{n2,t} & \ldots & \sigma_{nn,t} \end{bmatrix} \begin{bmatrix} \Delta Z_{1,t} \\ \Delta Z_{2,t} \\ \ldots \\ \Delta Z_{n,t} \end{bmatrix}, \tag{25}$$

such that the element of vector $\eta_\mathbf{t}$ occupying row i is:[15]

$$\eta_{\mathbf{t,i}} = \frac{1}{(2\pi)^{\frac{n+1}{2}} |\sigma_\mathbf{i}\sigma_\mathbf{i}^\mathbf{T}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left(\mathbf{LR_t} - \mathbf{MLR_i}\right)^\mathbf{T} \left(\sigma_\mathbf{i}\sigma_\mathbf{i}^\mathbf{T}\right)^{-1} \left(\mathbf{LR_t} - \mathbf{MLR_i}\right)\right], \tag{26}$$

[13] The demonstration of this formula can be found at Hamilton (1989).
[14] We assumed that any regime switch will take place at the end of period, as it is the standard in discrete-time regime switching models.
[15] Matrix $\sigma_\mathbf{i}$ is an n x n matrix built of $\sigma_{\mathbf{s,i}}$, like equation (25). The expression $|\sigma_\mathbf{i}{\sigma_\mathbf{i}}^T|$ denotes the determinant of matrix $\sigma_\mathbf{i}\sigma_\mathbf{i}^\mathbf{T}$.

where $\mathbf{LR_t}$ is an $(n+1) \times 1$ vector of the n risky assets observed log-returns at period t , *i.e.*, the left-hand side of equation (25). $\mathbf{MLR_i}$ stores the mean of log-returns conditioned by regime:

$$\mathbf{LR_t} = \begin{bmatrix} \ln \frac{S_{1,t+1}}{S_{1,t}} \\ \ln \frac{S_{2,t+1}}{S_{2,t}} \\ \dots \\ \ln \frac{S_{n,t+1}}{S_{n,t}} \end{bmatrix} \qquad \text{and} \qquad \mathbf{MLR_i} = \begin{bmatrix} \mu_{1,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{1k,i} \\ \mu_{2,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{2k,i} \\ \dots \\ \mu_{n,i} - \frac{1}{2}\sum_{k=1}^{n} \sigma^2_{nk,i} \end{bmatrix}. \tag{27}$$

E Estimation of Matrix σ_π

To estimate matrix σ_π, the parameter introduced by the CG model, we create an $n \times m$ matrix (denoted by $\mathbf{D_t}$) consisting of $\mathbf{D_{s,t}}$. Then we calculate the monthly time series for the drifts ($\mathbf{D_t}\pi_\mathbf{t}$) and volatility matrix ($\mathbf{V}\pi_\mathbf{t}$) as seen by the investor.[16] We are therefore able to estimate the discretized process below, which follows from equation (5):

$$\Delta \mathbf{Z^*_t} = \sigma_\mathbf{t}{}^{-1} \left(\mathbf{LR_t} - \mathbf{D_t}\pi_\mathbf{t}\right), \tag{28}$$

where $\mathbf{LR_t}$ is an $n \times 1$ vector of the n risky assets observed log-returns at period t. The matrix $\sigma_\mathbf{t}$ represents ($\mathbf{V}\pi_\mathbf{t}$), which is a square matrix of order n. We use equation (9) to write the discretization $(\Delta\pi_\mathbf{t} - \mu_{\pi,\mathbf{t}}) = \sigma_\pi \Delta \mathbf{Z^*_t}$ and store its left-hand time series in an $m \times T$ matrix denoted by $(\Delta\pi - \mu_\pi)$, as well as we store all increments $\Delta \mathbf{Z^*_t}$ in an $n \times T$ matrix denoted simply by $\Delta \mathbf{Z}^*$ (where T is the time series length). We finally obtain the desired estimation:

$$\sigma_\pi = (\Delta\pi - \mu_\pi)\, \Delta \mathbf{Z}^{*\mathbf{T}} \left(\Delta \mathbf{Z}^* \Delta \mathbf{Z}^{*\mathbf{T}}\right)^{-1}. \tag{29}$$

[16] To achieve a more robust methodology, we disregarded the initial 12 weekly observations. This is explained by the fact that the starting probabilities still had some effects over the filtered probabilities during the initial period, which revealed to be irrelevant after this starting date.

F Tables

Table 1: We show below the annualized absolute mean returns and volatilities of the indexes that were evaluated to represent the Brazilian asset classes. The observation windows were collected in weekly basis, ending in August 31st, 2018. IMA-G and IBrX-100 appear twice to be compared with the subsequent indexes whose observations only started later. Correlations originate from weekly data but are atemporal. When presented, correlations were computed in relation to the main asset of each class, respectively: CDI, IMA-G, and IBrX-100. The US stocks are represented by S&P 500 × PTAX.

Absolute Returns and Volatilities in the Single State Model				
Asset Classes	Observations Since	Mean Returns	Volatility	Correlation
Money				
CDI	Dec. 7th, 2001	12.90%	0.56%	-
LFT	Dec. 7th, 2001	12.96%	0.56%	99.99%
Fixed Income				
IMA-G	Dec. 7th, 2001	14.24%	2.82%	-
IRF-M	Dec. 7th, 2001	14.37%	2.70%	86.41%
IMA-G	Sep. 26th, 2003	13.24%	2.91%	-
IMA-B	Sep. 26th, 2003	14.89%	6.21%	95.70%
BR Stocks				
IBrX 100	Dec. 7th, 2001	18.27%	24.55%	-
Ibovespa	Dec. 7th, 2001	14.90%	26.28%	97.81%
IBrX 100	Sep. 9th, 2008	6.26%	24.33%	-
MLCX	Sep. 9th, 2008	6.68%	24.46%	99.92%
SMLL	Sep. 9th, 2008	6.24%	24.43%	85.87%
US Stocks				
S&P 500	Dec. 7th, 2001	7.10%	16.61%	-
PTAX	Dec. 7th, 2001	4.55%	15.66%	-
S&P 500 × PTAX	Dec. 7th, 2001	10.66%	16.93%	-

Table 2: Panel A reports the parameters and the optimal weights for the single state model (for an investor with $\gamma = 5$). The weights sum 100% when considering the allocation in the money asset class. Panel B presents the parameters for the four-state model. All parameters relate to risk premiums. The correlation matrices present the volatilities in their diagonals. The data were collected in weekly basis, and then annualized for presentation. The US stocks asset class was defined in Brazilian currency terms. The reported regimes and parameters denote the states of the Brazilian economy.

Panel A: Single State Model	Fixed Income	BR Stocks	US Stocks
Mean Risk Premiums	1.19%	4.77%	-1.97%
Correlation Matrix - Fixed Income	2.8%		
BR Stocks	32.0%	24.6%	
US Stocks	-20.6%	9.0%	16.9%
Optimal Weights	291.45%	5.32%	-4.84%

Panel B: Four State Model	Fixed Income	BR Stocks	US Stocks
Mean Risk Premiums - Regime 1 (Crash)	4.55%	-16.08%	-7.66%
Regime 2 (Bear)	-0.23%	-5.26%	21.44%
Regime 3 (Bull)	2.07%	13.77%	-11.38%
Regime 4 (Recovery)	-0.07%	25.26%	-21.55%
Correlation Matrix - *Regime 1 (Crash)*			
Fixed Income	3.1%		
BR Stocks	46.6%	46.8%	
US Stocks	-4.5%	4.6%	31.5%
Correlation Matrix - *Regime 2 (Bear)*			
Fixed Income	4.1%		
BR Stocks	43.1%	18.9%	
US Stocks	-37.8%	11.8%	14.8%
Correlation Matrix - *Regime 3 (Bull)*			
Fixed Income	1.2%		
BR Stocks	26.4%	21.5%	
US Stocks	-10.5%	2.4%	13.0%
Correlation Matrix - *Regime 4 (Recovery)*			
Fixed Income	0.3%		
BR Stocks	19.0%	20.9%	
US Stocks	-10.0%	0.4%	15.2%

Transition Probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1 (Crash)	88.89%	0.66%	10.09%	0.37%
Regime 2 (Bear)	1.00%	97.50%	1.25%	0.25%
Regime 3 (Bull)	0.50%	2.00%	97.00%	0.50%
Regime 4 (Recovery)	2.00%	1.50%	1.50%	95.00%

Table 3: We show below the estimated volatility matrix for the probability processes when there are four regimes with the three risky assets described on the text. The estimation procedure is explained in appendix E. The table also reports the ergodic (unconditional) probabilities and the average duration of each regime. We used weekly observations from April 5^{th}, 2002 to August 31^{st}, 2018

	Z_1	Z_2	Z_3	Ergodic Probabilities	Average Duration
Regime 1	1.17%	-0.97%	-0.36%	7.0%	9 weeks
Regime 2	-0.51%	0.16%	2.04%	41.6%	40 weeks
Regime 3	-0.21%	0.49%	-1.56%	44.4%	33 weeks
Regime 4	-0.45%	0.32%	-0.12%	7.0%	20 weeks

Table 4: We show the mean returns of the assets during the regimes. Here, we perform a historical analysis (with real data), not the implementation of an investment strategy. Thus, we were able to use the smoothed probabilities, which are the best information to infer the most likely regime at each point of observation. We denoted the absolute returns of each asset according to these most likely regimes, to calculate the mean returns. We used weekly observations from December 7^{th}, 2001 to August 31^{st}, 2018, and below we present annualized data. The US stocks are represented by S&P 500 $\times$ PTAX.

Absolute Mean Returns in the Four State Model				
Asset Classes	Regime 1 (Crash)	Regime 2 (Bear)	Regime 3 (Bull)	Regime 4 (Recovery)
Money				
CDI	15.04%	10.57%	12.30%	19.25%
Fixed Income				
IMA-G	20.35%	10.38%	14.61%	19.18%
BR Stocks				
IBrX 100	-9.73%	4.86%	26.44%	58.07%
US Stocks				
S&P 500	-34.33%	13.53%	11.92%	12.98%
PTAX	71.31%	19.56%	-10.50%	-16.53%
S&P 500 $\times$ PTAX	6.77%	35.21%	-0.87%	-6.16%

Table 5: We show below the comparison of the dynamic portfolio strategies, and also the accuracy of the solution obtained through the CG model. The portfolios were unconstrained (short selling allowed). The simulation results denote the optimal solution, while the CG Model results refer to the suboptimal solutions obtained through the approximation. The wealth equivalent loss is due to the suboptimal solution in comparison to the optimal solution. The results are reported for different horizons in weeks, as we worked with weekly observations. Only the wealth equivalent loss is shown in yearly basis. We illustrate this comparison with the next period regime being either known or unknown by the investor. The former denotes the scenario where the investor infers the next period regime through the filtered probabilities. For this comparison, we estimated the cases where there are 100% chance (according to the filtered probabilities) of each of the four regimes to occur. In the other scenario, the investor has no access to the filtered probabilities, then the next period regime is inferred using the ergodic probabilities.

Horizon (in weeks)	Dynamic Portfolio Strategy								Wealth Equivalent Loss (per year)
	Simulation Results				CG Model Results				
	Money	Fixed Income	BR Stocks	US Stocks	Money	Fixed Income	BR Stocks	US Stocks	
Known Next Period Regime 1 (Crash)									
13	-1122.5%	1287.5%	-57.5%	-7.5%	-1127.3%	1284.7%	-54.5%	-2.8%	0.00%
26	-1197.5%	1362.5%	-57.5%	-7.5%	-1124.8%	1281.0%	-54.3%	-1.9%	0.13%
52	-1147.5%	1312.5%	-57.5%	-7.5%	-1123.9%	1279.7%	-54.2%	-1.6%	0.28%
130	-1110.0%	1275.0%	-57.5%	-7.5%	-1123.8%	1279.5%	-54.2%	-1.5%	0.38%
260	-1122.5%	1287.5%	-57.5%	-7.5%	-1123.8%	1279.6%	-54.2%	-1.5%	0.10%
Known Next Period Regime 2 (Bear)									
13	-400.0%	335.0%	-47.5%	212.5%	-397.7%	344.0%	-47.5%	201.2%	0.09%
26	-400.0%	347.5%	-47.5%	200.0%	-392.0%	340.5%	-47.2%	198.7%	0.00%
52	-412.5%	335.0%	-35.0%	212.5%	-390.3%	339.5%	-47.2%	198.0%	0.15%
130	-412.5%	347.5%	-47.5%	212.5%	-390.2%	339.4%	-47.1%	197.9%	0.00%
260	-425.0%	372.5%	-60.0%	212.5%	-390.2%	339.4%	-47.1%	197.9%	0.09%
Known Next Period Regime 3 (Bull)									
13	-2395.0%	2587.5%	40.0%	-132.5%	-2396.5%	2587.5%	42.6%	-133.6%	0.00%
26	-2470.0%	2662.5%	40.0%	-132.5%	-2395.2%	2584.1%	42.5%	-131.4%	0.02%
52	-2407.5%	2600.0%	40.0%	-132.5%	-2394.1%	2582.4%	42.5%	-130.7%	0.00%
130	-2320.0%	2525.0%	40.0%	-145.0%	-2393.9%	2582.0%	42.5%	-130.6%	0.06%
260	-2382.5%	2587.5%	40.0%	-145.0%	-2393.9%	2582.0%	42.5%	-130.6%	0.10%
Known Next Period Regime 4 (Recovery)									
13	4992.5%	-4800.0%	145.0%	-237.5%	4575.4%	-4376.5%	135.2%	-234.2%	0.21%
26	4905.0%	-4687.5%	132.5%	-250.0%	4579.6%	-4383.8%	135.0%	-230.9%	0.31%
52	4267.5%	-4062.5%	132.5%	-237.5%	4587.7%	-4392.9%	135.0%	-229.8%	0.21%
130	4330.0%	-4125.0%	132.5%	-237.5%	4589.3%	-4394.7%	135.1%	-229.7%	0.09%
260	4617.5%	-4400.0%	132.5%	-250.0%	4589.4%	-4394.8%	135.1%	-229.7%	0.31%
Unknown Next Period Regime: Ergodic Probabilities									
13	-217.5%	325.0%	-7.5%	0.0%	-315.8%	400.3%	-2.6%	18.1%	0.31%
26	-180.0%	250.0%	5.0%	25.0%	-314.4%	398.4%	-2.3%	18.3%	0.28%
52	-142.5%	225.0%	-7.5%	25.0%	-313.8%	397.6%	-2.3%	18.4%	0.74%
130	-130.0%	225.0%	-20.0%	25.0%	-313.7%	397.5%	-2.2%	18.4%	1.90%
260	-130.0%	225.0%	-20.0%	25.0%	-313.7%	397.5%	-2.2%	18.4%	3.17%

Table 6: We show below the results obtained through the CG model and the performances of the main Brazilian benchmarks within a 52-week moving window, in terms of: risk premium, volatilities and absolute returns (only for the CG model). The portfolio was unconstrained (short selling allowed). The allocation was computed for a 5-year horizon investor with relative risk aversion of $\gamma = 5$, who inferred the next period regime through the filtered probabilities. For every point in time, the filtered probabilities were estimated using only the information available before time $\mathtt{t}$. The US stocks asset class was defined in Brazilian currency terms.

Performances within 52-Week Moving Windows									
	Risk Premium				Volatility				Absolute Returns
Dates	Fixed Income	BR Stocks	US Stocks	CG Model	Fixed Income	BR Stocks	US Stocks	CG Model	CG Model
Last Week of the Year									
2002-12-27	-0.35%	-12.20%	-3.29%	24.24%	0.27%	4.30%	4.08%	3.98%	47.83%
2003-12-26	4.12%	44.02%	-16.83%	153.35%	0.09%	2.65%	2.44%	4.12%	212.24%
2004-12-31	0.75%	10.04%	-13.40%	58.53%	0.08%	3.24%	1.61%	3.83%	84.06%
2005-12-30	-0.67%	15.40%	-23.67%	114.31%	0.04%	3.12%	2.00%	4.53%	155.03%
2006-12-29	2.18%	18.29%	-9.78%	95.72%	0.15%	3.27%	1.30%	4.31%	125.13%
2007-12-28	0.71%	32.26%	-22.73%	22.17%	0.26%	3.54%	1.73%	4.02%	36.55%
2008-12-26	0.12%	-49.23%	-29.90%	-12.55%	0.43%	6.67%	3.14%	6.54%	-1.84%
2009-12-25	3.04%	58.19%	-12.83%	65.00%	0.18%	3.55%	2.84%	4.18%	81.33%
2010-12-31	2.94%	-6.50%	-1.66%	64.28%	0.19%	2.64%	1.88%	4.43%	80.30%
2011-12-30	1.84%	-20.60%	0.88%	19.51%	0.21%	2.96%	2.55%	4.27%	33.37%
2012-12-28	8.60%	2.94%	12.10%	79.67%	0.41%	2.10%	1.58%	2.87%	94.71%
2013-12-27	-8.82%	-10.87%	40.02%	39.15%	0.63%	2.01%	1.85%	3.21%	50.30%
2014-12-26	1.28%	-11.51%	16.67%	37.54%	0.62%	3.15%	2.17%	3.42%	52.33%
2015-12-25	-3.30%	-21.47%	28.55%	34.06%	0.64%	2.96%	2.51%	4.20%	51.69%
2016-12-30	6.14%	19.91%	-19.80%	33.50%	0.49%	3.59%	2.63%	4.84%	52.19%
2017-12-29	2.63%	16.03%	10.27%	-49.24%	0.47%	2.29%	1.44%	8.23%	-44.20%
Last Observation									
2018-08-31	-0.63%	-0.89%	44.79%	-7.89%	0.35%	2.55%	2.05%	4.43%	-1.63%
CG Model Highest Performance									
2006-03-17	0.93%	22.24%	-27.84%	192.72%	0.10%	3.11%	1.93%	4.71%	247.70%
CG Model Lowest Performance									
2017-05-26	1.84%	15.83%	-8.34%	-59.99%	0.53%	2.59%	2.17%	8.64%	-54.70%
CG Model Average Performance									
Since Inception	1.43%	5.21%	-2.53%	50.37%	0.33%	3.23%	2.20%	4.42%	71.05%
Since 2012	1.20%	-2.01%	14.03%	29.26%	0.52%	2.74%	2.12%	4.44%	43.14%

G Figures

Figure 1: We show the filtered probabilities of being on each of the states. For every point in time, these probabilities were realistically estimated using only the information available before time t. We began our filter on December 7^{th}, 2001 and after only 12 weeks, the probabilities were not sensitive to the starting point. The window of observations reported below is weekly from April 5^{th}, 2002 to August 31^{st}, 2018.

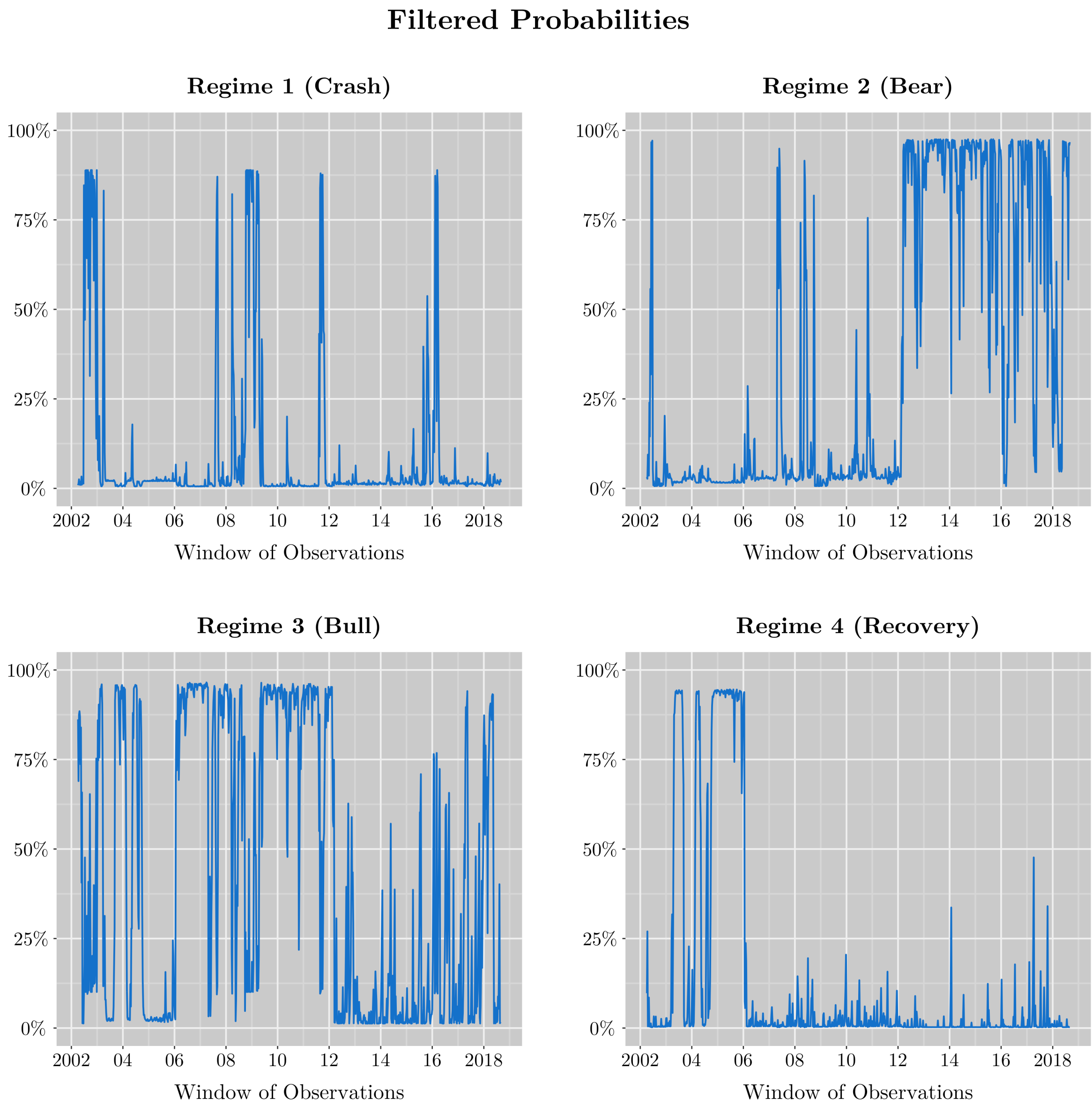

Figure 2: We grouped the filtered probabilities of being on each of the states in one figure, to convey the outlook of the regime switching according to this process. For every point in time, these probabilities were realistically estimated using only the information available before time t. We began our filter on December 7th, 2001 and after only 12 weeks, the probabilities were not sensitive to the starting point. The window of observations reported below is weekly from April 5th, 2002 to August 31st, 2018.

Filtered Probabilities

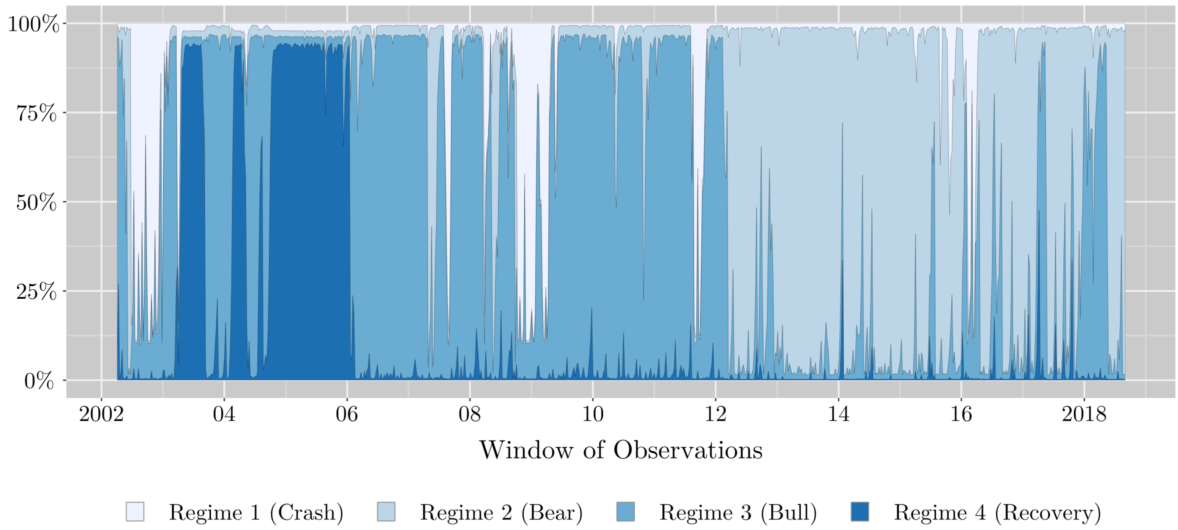

Figure 3: We grouped the smoothed probabilities of being on each of the states in one figure, to convey the outlook of the regime switching according to this process. These are post-calculated probabilities using the whole set of information through all the data sample, which goes from December 7th, 2001 till August 31st, 2018. The window of observations reported below is weekly from April 5th, 2002 to August 31st, 2018.

Smoothed Probabilities

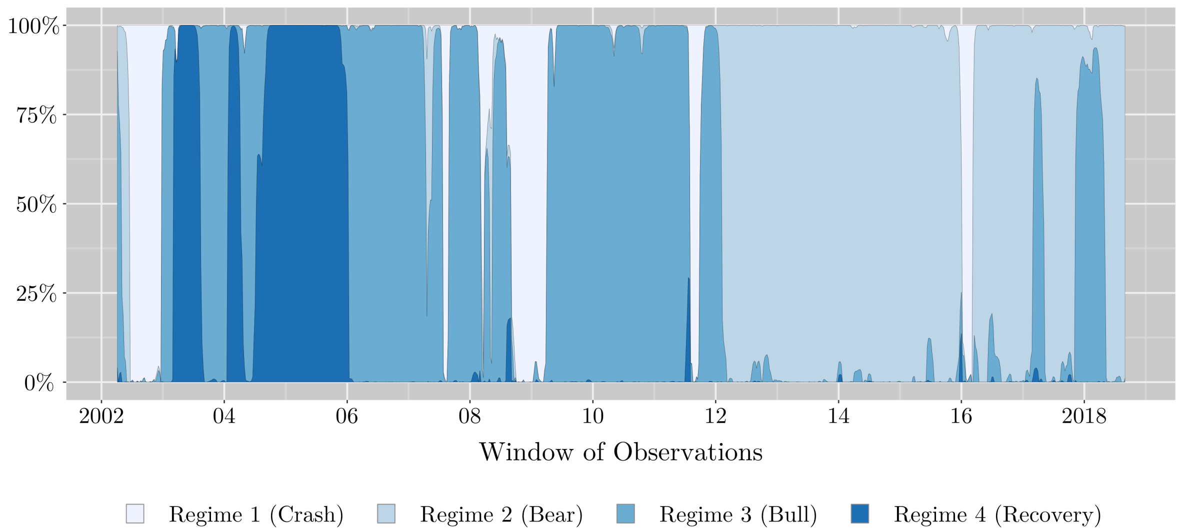

Figure 4: We show the the most likely regime according to the smoothed probabilities presented together with the accumulated risk premium of the three risky assets. These are post-calculated probabilities using the whole set of information through all the data sample, which goes from December 7th, 2001 till August 31st, 2018. The window of observations reported below is weekly from April 5th, 2002 to August 31st, 2018. The US stocks asset class was defined in Brazilian currency terms.

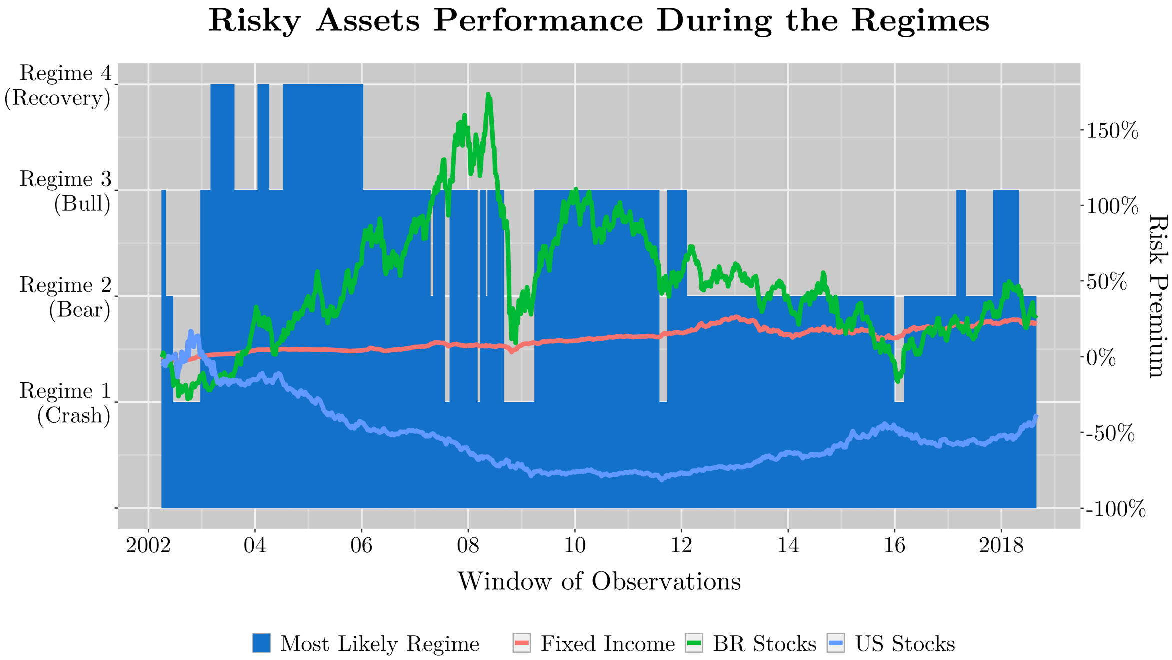

Figure 5: We show the frequency distribution of the filtered probabilities. At each observation point, we classified the probabilities according to their size in relation to the other probabilities (this classification combines all regimes). At each observation point, the first largest probability denotes the most likely regime to occur, whereas the smallest probability is the less likely regime to occur. The window of observations considered below is weekly from April 5th, 2002 to August 31st, 2018.

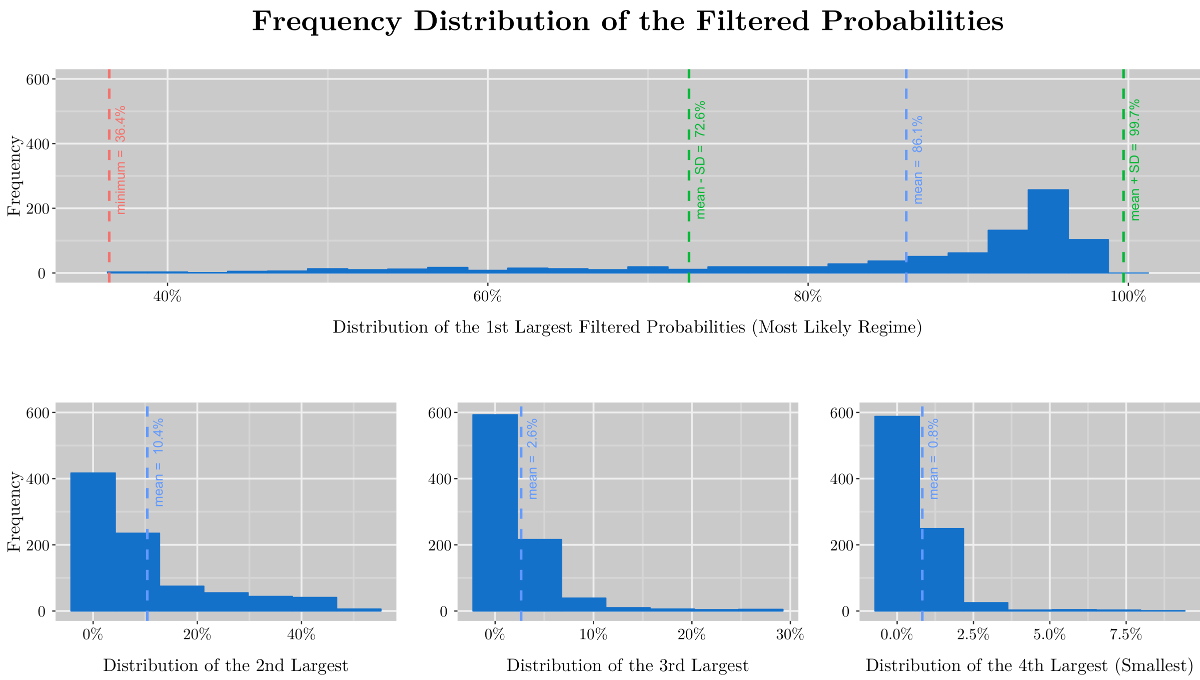

Figure 6: We show the dynamic strategy variation for a 5-year horizon investor, whose relative risk aversion was $\gamma = 5$. The portfolio was unconstrained (short selling allowed) and we considered the investor using the filtered probabilities to infer the next period regime. For every point in time, these probabilities were realistically estimated using only the information available before time t. We began our filter on December 7^{th}, 2001 and after only 12 weeks, the probabilities were not sensitive to the starting point. The window of observations reported below is weekly from April 5^{th}, 2002 to August 31^{st}, 2018. The US stocks asset class was defined in Brazilian currency terms.