LOCAL ESTIMATION OF DYNAMIC COPULA MODELS

BEATRIZ V. M. MENDES
IM/COPPEAD Graduate School of Business
Federal University at Rio de Janeiro, Brazil
beatriz@im.ufrj.br

EDUARDO F. L. DE MELO
Mathematics and Statistics Institute
State University of Rio de Janeiro, Brazil

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It has been empirically verified that the strength of dependence in stock markets usually rises with volatility. In this paper we exploit this stylized fact combined with local maximum likelihood estimation of copula models to analyze the dynamic joint behavior of series of financial log returns. Explanatory variables based on the estimated GARCH volatilities are considered as potential regressors for explaining the dynamics in the copula parameters. The proposed model can assess and discriminate how much of the strength of dependence is due just to the time-varying volatility. The final local-parametric estimates may be used to compute risk measures, to simulate portfolio behavior, and so on. We illustrate our methods using two American indexes. Results indicate that volatility does affect the strength of dependence. The in-sample Value-at-Risk based on the dynamic model outperforms those based on the empirical estimates.

Keywords: Copulas; local maximum likelihood estimation; GARCH models; Value-at-Risk.

1. Introduction

Estimating and simulating the evolution of many products in finance requires accurate modeling of the dependence between the underlying assets. To capture all possible different types of linear and non-linear association among these components, one needs copula models. A copula is a multivariate distribution with standard uniform marginal distributions. Modeling using copulas allows for factoring the multivariate distribution into its marginal univariate distributions and a copula. In practice, this fact simplifies both the specification of the multivariate distribution and its estimation.

*Corresponding author.
In finance, the estimation of time-varying marginal moments may be accurately carried on by means of some combination of ARFIMA and FIGARCH type models [1–3, 12]. However, residual dynamic is sometimes found in the dependence structure and calls for proper modeling. Empirical evidence of correlation breakdowns had been reported by Longin and Solnik [22], who noted that the level of correlation may be related to level of market volatility. However, dynamic models were not used. Instead, extreme value multivariate models to model dependence were considered. All initial attempts to multivariate dynamic modeling in finance were based on multivariate GARCH models [13, 35]. The estimation of multivariate GARCH models however still demands further care. An important contribution in this area is Engle [13]. The dynamic conditional correlation (DCC) model combines the flexibility of the univariate GARCH models with parametric models for the correlations. However, to combine good marginal fits and to measure dependence through other dependence measures beyond correlations one needs dynamic copula models combined with univariate GARCH modeling.

Patton [30] introduced the conditional copula in the bivariate case. Modeling exchange rates, he assumed a bivariate Gaussian conditional copula with the correlation coefficient following an ARMA model based on a logistic transformation. He considered also structural breaks and asymmetric copulas. This paper was later circulated as Patton [31] and finally published as [29].

Rockinger and Jondeau [32] assumed a parametric copula conditional to the position of past joint observations in the unit square, combined with previous marginal estimation of GARCH-type models with time-varying skewness and kurtosis. Using the Hansen’s [20] generalized t-student as the error distribution for the GARCH models and the Plackett’s copula, they provided empirical evidence that the dependency between financial returns may evolve through time.

Fermanian and Wegkamp [17] introduced the concept of pseudo-copulas. They showed that the copula models defined in [30, 32] are all pseudo-copulas. They proposed a nonparametric estimator of the conditional pseudo-copulas, derived its normal asymptotic distribution, and built up a goodness of fit test statistics. The concept of pseudo-copulas formalized and unified previous attempts in the direction of modeling time varying dependence structures using copulas.

Cherubini et al. [5] fitted a GARCH(1,1) model to the margins and modeled the dependence structure with a Gaussian copula with a time-varying correlation coefficient. In order to keep the correlation coefficient between −1 and 1, a modified logistic transformation of the copula parameter was considered. Dias and Embrechts [9] applied univariate GARCH models and the t-copula model with time varying correlation to high frequency data.

Mendes [24] extended Rockinger and Jondeau [32] model and assumed a parametric pseudo-copula conditional to the position of lag 1 past joint and lag 2 past joint observations in the unit square, combined with FIGARCH estimation of the univariate distributions of the log-returns. The time-varying Value-at-Risk (VaR) was computed and compared to static copula models.
Van Den Goorbergh et al. [37] studied the behavior of bivariate option prices when the dependence structure of the underlying financial assets follows a dynamic copula model. The copula parameter is estimated based on a rolling window and trends are detected as functions of past volatilities.

In this paper, to assess the full dynamic dependence structure existing among assets we apply local maximum likelihood estimation to copula parameters. This simple model is improved with the addition of explanatory variables and a time-varying slope. As suggested by empirical evidence, we consider as regressors the current and past volatilities. This would enhance the predictive power of the copula model. To illustrate, we compute an equally weighted portfolio and estimate the one-step ahead VaR, combining a forecast for the copula parameter and a forecast for the marginal conditional means and variances.

The remaining of this paper is organized as follows. In Sec. 2, we provide a brief review of conditional copula definitions and estimation methods. In Sec. 3, we define the local maximum likelihood estimates, discuss inference and diagnostics and propose the extension based on volatilities as explanatory variables. In Sec. 4, we use a bivariate data set of daily log-returns on SP500 and Nasdaq to illustrate the estimation steps of models proposed. In Sec. 5, we provide another application using maxima and minima from the daily returns. In Sec. 6, we discuss the results and provide some concluding remarks.

2. Copulas and Classical Estimation

To simplify the notation, from now on we set \( d = 2 \) even though the inference methods in the paper are intended for and work for dimensions \( d \geq 2 \).

Consider a stationary process \((X_{1,t}, X_{2,t})_{t \in \mathbb{Z}}\). In the case the joint law of \((X_{1,t}, X_{2,t})\) is independent of \( t \), the dependence structure of \((X_{1}, X_{2})\) is given by its (constant) copula \( C \). If \((X_{1}, X_{2})\) is a continuous random vector with joint cdf \( F \) and marginal cdfs \( F_1 \) and \( F_2 \), then there is a unique copula \( C \) pertaining to \( F \), defined on \([0,1]^2\) such that

\[
C(F_1(x_1), F_2(x_2)) = F(x_1, x_2)
\]

holds for any \((x_1, x_2) \in \mathbb{R}^2\).

The copula \( C \) summarizes the dependence structure of \( F \), independently of the specification of the marginal distributions. It is invariant under strictly increasing transformations of \((X_{1}, X_{2})\) being therefore very convenient for studying dependence structure with tools which are scale-free.

Now, let \((X_{1}, X_{2})\) be a continuous random vector from \((\Omega, \mathcal{A}, P) \to \mathbb{R}^2\), where \( \mathcal{A} \) is the sigma algebra generated by all past information, and consider the sample \((x_{1,1}, x_{2,1}), \ldots, (x_{1,N}, x_{2,N})\). In the time series context

\[
\mathcal{A}_t = \sigma \{x_{1,t-1}, x_{2,t-1}, x_{1,t-2}, x_{2,t-2}, \ldots\} \quad t = 1, \ldots, N
\]

represents the past information up to time \( t \).
Patton [30] first extended the Sklar’s theorem

\[ F(x_{1,t}, x_{2,t} | \mathcal{A}_t) = C_t(F_{1,t}(x_{1,t} | \mathcal{A}_t), F_{2,t}(x_{2,t} | \mathcal{A}_t) | \mathcal{A}_t), \]

(2.1)

where \( C_t \) is a copula at times \( t \). Note that in general the sample data matrix may not represent \( T \) observations of the same joint distribution. The same conditioning set for each marginal and for the conditional copula guarantees that each transformed variable is independent of the information in the conditioning set of its marginal distribution. A time dependent copula may not satisfy all properties of a copula (see concept of pseudo-copulas in Fermanian and Wegkamp [17] and Nelsen [28]).

Parametric estimation of copulas may be accomplished in two steps. In the first step the univariate marginal distributions are estimated, the \( \hat{F}_{i,t} \), \( i = 1, 2 \), are used to compute the pseudo uniform \((0, 1)\) data, which are used in the second step to estimate the copula parameters. The marginal estimation in the first step may be based on a parametric model or just computed using the empirical distribution, a method actually preferred by some authors to avoid misspecification problems [18].

In the context of independent and identically distributed observations, besides the maximum likelihood method [21], the copula parameters estimates in the second step may be obtained by robust and minimum distance estimators [26, 36], or semi parametrically ([38], where a new semi parametric copula family is proposed and estimated using a penalized constrained least squares method). In the context of univariate time series a related paper is Fermanian and Scaillet [16].

Dynamic maximum likelihood estimation of copula models are usually carried on by assuming some autoregressive model, as in Patton [29] or in Cherubini et al. [5]. To the best of our knowledge, dynamic copula models based on local maximum likelihood estimation have not yet been proposed.

3. Local Maximum Likelihood Estimation of Copula Models

In this section we develop our methodology for estimating copulas dynamically. Inference is carried on in two steps and is build on the local maximum likelihood estimates for copula parameters. Consider \( \{U_{t1}, V_{t1}\}, \ldots, \{U_{tN}, V_{tN}\} \) a possibly weakly dependent bivariate time series. It is assumed that at time \( t_j \) the continuous distribution of \( \{U_{tj}, V_{tj}\} \) has conditional copula density \( c(u_{tj}, v_{tj}, \theta | \mathcal{A}_t) \), where the vector parameter \( \theta_j \in \mathbb{R}^k \), \( k \geq 1 \), is a smooth function of \( t_j \).

The local-constant estimator \( \hat{\theta}_j \) is the value \( \theta \) in the copula parameter space maximizing

\[ \sum_{j=1}^{N} K \left( \frac{t_j - \tau_j}{h} \right) \log(c(u_{j}, v_{j}, \theta | \mathcal{A}_t)), \]

(3.1)

where \( K \) is a symmetric positive kernel weight function which at time \( t_j \) gives weights for times \( t_J, J = 1, \ldots, N \), according to \( K(\frac{t_j - t_J}{h}) \), and where \( h > 0 \) is the temporal bandwidth. One appealing characteristic of this method is its simplicity: it can be applied to any continuous copula family.
According to several authors \[7\], among others, in practical applications the choice of the kernel function is not usually critical. In this paper we use the standard choice of the normal density with zero mean and standard deviation \(h\). The choice of the bandwidth \(h\) is crucial and affects both the estimator bias and variance. When choosing \(h\) subjectively, one may use prior information and check graphically the validity of fitted parameters values produced by different bandwidths. Other approaches consider choosing \(h\) automatically, as the fully-automatic bandwidth selection procedure of Fan and Gijbels \[14\]; or consider minimizing some squared loss function, as in Müller \textit{et al.} \[27\]. In practice, the choice of \(h\) is based on the combination of preliminary experimentation, expertise in the field of application, and on the visual assessment of the bias-variance trade-off, which should be done for several \(h\) values. The final estimates should have small (local) bias and still capture the dynamics in the dependence structure.

Whatever the method for choosing the bandwidth \(h\), the \(h\) value chosen must provide a reasonable fit to the data. We propose here to use local pp-plots based on the estimated copula and on the empirical copula. For a fixed time \(t\), let \(\hat{\theta}_t\) be the corresponding local estimate. We select (a few) times \(t\) over the whole period covered and collect the data inside the window \([t-h, t+h]\). For the window data we compute the empirical copula and the estimated copula based on \(\hat{\theta}_t\) (cumulative distributions). As pointed out in Hall and Tajvidi \[19\], to reduce the systematic bias in the estimators, a smaller value than the optimal for \(h\) could be used to construct the pp-plots. This is also supported by the fact that the data points closer to the edges of the interval receive less weight than those near to \(t\).

The asymptotic normal distribution of the local maximum likelihood estimates for copulas may be easily derived as an extention of Theorem 5.1 of Hall and Tajvidi \[19\]. As in any other application of local maximum likelihood methods, under weak dependence the estimators first order biases remain unchanged, even though the asymptotic variances may differ. References for local likelihood inference are Tibshirani and Hastie \[34\], Fan and Gijbels \[14\], and several papers published with applications in different areas.

The uncertainty in the local maximum likelihood estimates may also be quantified through bootstrap samples (see \[7\]). According to Davison and Hinkley \[6\], under weak assumptions, if the number of bootstrap samples is sufficiently large, then the empirical variability within the bootstrapped estimates shall give a good approximation for the uncertainty of the local estimator. Nevertheless, due to the presence of bias in the estimates, the construction of bootstrap confidence intervals about parameter estimates becomes complicated. Deciccio and Romano \[8\] proposed methods for constructing confidence intervals that account for bias in the estimates. Bowman and Azzalini \[4\] call the attention to the fact that confidence intervals may be constructed without correcting for bias, although, in this case, the variability bands cover \(E(\hat{\theta}(t))\) rather than \(\hat{\theta}(t)\).
A straightforward generalization of the local constant estimator (3.1) is the local-linear estimator, where the copula parameter is formulated as a local polynomial function of time. In this paper we exploit the stylized fact in finance that the strength of (dynamic) dependence may depend on volatility ([11, 22, 25], among others), and propose a variation of the local linear estimator where the estimated conditional volatilities are used as explanatory variables. The local-linear estimator regressed on explanatory variables $X$ is defined as the value

$$
\hat{\theta}_j = \hat{\theta}_0 + \hat{\beta}X,
$$

which maximizes (3.1) with respect to all $(\theta_0, \beta)$ in the parameter space and subject to restrictions on the final estimate $\theta$, and where $X$ is some function of previous volatilities. To guarantee that the final estimates are within their corresponding parameters space — for example, in the case of the correlation coefficient, $\rho$ should be in $[-1, +1]$ — the algorithm set restrictions or consider parametric transformations.

To illustrate the usefulness of the models proposed we analyze in the following two sections a 5-years sample composed by 1304 pairs of daily log-returns on the SP500 and Nasdaq from June 3, 2002 to May 31, 2007, obtained from Datastream.

4. Dynamic Copula Local Estimation of Joint Daily Returns

In almost all applications using financial returns, the copula data $(U_{t_1}, V_{t_1}), \ldots, (U_{t_N}, V_{t_N})$ are obtained (in the first step) as the empirical or model based probability integral transformation of the uncorrelated standardized residuals from GARCH marginal fits to the log-returns. As we shall see, residual temporal dependence in copula data may still exist, and will be captured non parametrically by the local estimates plus a parametric specification based on previous volatilities.

We fit by maximum likelihood GARCH-type models [1, 2, 12] with Gaussian innovations to each margin. As required by the conditional copulas theory and pointed out in Patton [29, 30], we assume that the marginal models are specified conditional on a common past information set, to guarantee that the joint conditional distribution is a copula. In practice this means that we are assuming there will be no contagion between the marginal distributions.

Table 1 shows parameters estimates and standard errors of the best models found for each margin considering just short memory in volatility (upper panel), and also long memory in volatility (lower panel). The estimated GARCH-volatilities for both indexes (SP500 in solid line and Nasdaq in dotted line) are shown in the second row of Fig. 1. The upper plot of Fig. 1 shows the time series plot of the daily returns on Nasdaq. According to the AIC criterion the best fits for this data set are based on the long-memory GARCH models. However we continue our analyses considering the residuals from the four fits.

The estimated standardized returns innovations are then ranked to obtain the pseudo uniform$(0,1)$ data. They form the support set for the empirical
Table 1. Quasi-maximum likelihood estimates (standard errors) of the GARCH and FIEGARCH models fitted to marginal index return data.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>S&amp;P500</th>
<th>Nasdaq</th>
</tr>
</thead>
<tbody>
<tr>
<td>MA(1)</td>
<td>-0.0593 (0.0311)</td>
<td>-0.0467 (0.0301)</td>
</tr>
<tr>
<td>( \alpha_0 )</td>
<td>0.0099 (0.0025)</td>
<td>0.0067 (0.0030)</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>0.0514 (0.0094)</td>
<td>0.0362 (0.0073)</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>0.9327 (0.0120)</td>
<td>0.9580 (0.0079)</td>
</tr>
<tr>
<td>AIC Criterion</td>
<td>3128.709</td>
<td>4223.170</td>
</tr>
<tr>
<td>MA(1)</td>
<td>-0.0603 (0.0245)</td>
<td>-0.0481 (0.0250)</td>
</tr>
<tr>
<td>( \alpha_0 )</td>
<td>-0.0396 (0.0103)</td>
<td>-0.0292 (0.0094)</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-0.2196 (0.0457)</td>
<td>-0.1398 (0.0533)</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>0.2660 (0.0487)</td>
<td>0.1770 (0.0583)</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>0.8667 (0.0462)</td>
<td>0.7547 (0.1141)</td>
</tr>
<tr>
<td>Leverage Term</td>
<td>-0.1962 (0.0272)</td>
<td>-0.1308 (0.0310)</td>
</tr>
<tr>
<td>Fraction ( d )</td>
<td>0.4656 (0.0629)</td>
<td>0.7144 (0.0795)</td>
</tr>
<tr>
<td>AIC Criterion</td>
<td>3060.134</td>
<td>4194.599</td>
</tr>
</tbody>
</table>

Fig. 1. Time series plot of the daily returns on Nasdaq (upper panel), and the estimated GARCH-volatilities (lower panel) for both indexes (S&P500 in solid line and Nasdaq in dotted line).
Fig. 2. First row: Scatter plot of the daily returns (left), and the support set for the empirical copula of the standardized GARCH innovations (right). Second row: Simulated data from the fitted t-copula (8 df) (left), and the pp-plot based on the empirical copula and the fitted t-copula.

copula\(^1\) of the standardized GARCH innovations, which estimates their unconditional joint distribution. Their scatter plot is shown at the right hand side of the first row of Fig. 2, along with the scatter plot of the daily returns (at the left hand side). We observe a strong positive association as well as some indication of tail

\(^1\)Definition of empirical copula: Let \((X_{1,t}, X_{2,t})\), \(t = 1, \ldots, N\), be \(N\) independent copies of \((X_1, X_2)\) with joint cdf \(F\), continuous marginal cdf’s \(F_1\) and \(F_2\), and possessing copula \(C\). The bivariate empirical distribution function is given by

\[
F_N(x, y) = \frac{1}{N} \sum_{t=1}^{N} \mathbb{I}_{\{X_{1,t} \leq x, X_{2,t} \leq y\}}, \quad -\infty < x, y < +\infty,
\]

where \(\mathbb{I}_{\{A\}}\) is the indicator function of event \(A\). Its associated marginal empirical distribution functions \(F_{i,N}(x)\), \(i = 1, 2\), are defined by \(F_{1,N} = F_N(x, +\infty)\) and \(F_{2,N}(y) = F_N(+\infty, y)\). Let \(F_{i,N}^{-}\) represent the generalized inverse of \(F_{i,N}\), that is, \(F_{i,N}^{-}(u) = \sup\{t \in \mathbb{R} : F_{i,N}(t) \leq u, 0 \leq u \leq 1\}\). The empirical copula function \([15]\) \(\tilde{C}\) is defined by

\[
\tilde{C}(u, v) = F_N(F_{1,N}^{-}(u), F_{2,N}^{-}(v)), \quad 0 \leq u, v \leq 1.
\]
dependence. The sample estimates of Kendall’s tau based on the ranked data from the GARCH and the FIEGARCH fits are, respectively, 0.7067 and 0.7004.

Our local estimation procedure assumes that the copula family is fixed throughout the period being analyzed, being time-varying just the parameters. A more complex model could assume that shape and parameters would change with time. The decision on which copula family is to be used in the local estimation is based on the best constant copula fit. To account for the features observed in the scatter plots in the first row of Fig. 2, and to cover the empirical facts usually observed for financial returns, we fit 5 copula families to the data assuming static dependence. They are the BB7 (terminology in [21]), which models different upper and lower tail dependence; the Gumbel which models just upper tail dependence, the Normal (models asymptotic independence), the Normal.mix (terminology in SPlus, a mixture of two Gaussian distributions); and the t (symmetric upper and lower tail dependence). Finding the best copula for the data at hand is actually an important and difficult problem [10].

Their log-likelihoods for the GARCH model based data were respectively: 966.17 (BB7, two parameters); 987.150 (Gumbel, one parameter); 1015.00 (Normal, one parameter); 1025.15 (Normal-mix, three parameters), and 1024.38 (t with 8 degrees of freedom (df), two parameters). Despite its flexibility, for this data set the BB7 copula did not provide a good fit. The software reported a relative function convergence and standard errors could not be estimated. All parameters estimates for the other copulas were highly significant. By penalizing for the number of parameters we finally choose the t-copula (8 df) as the best fit. The t-copula also turned out as the best fit when using the FIEGARCH residuals. Thus we continue the analyzes assuming the t-copula. The corresponding global maximum likelihood estimates of the correlation coefficient \( \hat{\rho} \) are 0.8906 and 0.8850.

Figure 2 compares the standardized data (right-upper plot) and 1304 bivariate points simulated from the fitted t copula with 8 df (left-bottom plot). They look very similar. To graphically assess goodness of fit we examine the pp-plot shown in the right-hand side of second row of Fig. 2. For each data point we compute the cumulative probability based on the empirical copula and on the estimated constant copula.

To fit the local model we firstly choose \( h \). In finance, it seems reasonable to choose the bandwidth “subjectively”, based upon on a prior information about the persistence of news or trends in the market. According to expertise information, effect of news would persist at most for 3–4 weeks. We visually assess the bias-variance trade off by examining the behavior of the \( \hat{\rho}'s \) paths obtained from different \( h \)s. After some experimentation, we set \( h \) in the range 15–20 business days. This is in line with the rolling window size of 2 months considered by Van Den Goorbergh et al. [37]. The range for the degrees of freedom considered was 5 to 12. The final best fit was based on \( h = 20 \) and, since there was not too much variation in the degrees of freedom (average was 8), from now on we keep it fixed and equal to 8.
The volatility model specification (just short memory or short and long memory) had a slight impact on the paths of local $\rho$ estimates. These paths showed similar fluctuations, differing in magnitude during short periods of time, being this more evident at the (right) end of the series.

As explained in Sec. 3, to assess goodness of fit of the local estimates we apply an extension to the copula case of the probability plot. Figure 3 (based on the GARCH residuals) shows the pp-plots and confirm the good fits for the selected time points $t$ corresponding to the percentuals $\{0.1, 0.2, \ldots, 0.9\}$ of the whole period of 1304 days. According to this figure, the local maximum likelihood estimates were able to capture the dynamics in the copula parameter.

The upper panel of Fig. 5 shows the evolution over time of the local constant estimates $\hat{\rho}_j$ during the period considered. We look for trends or any other functional form of time. At this point, if a trend over the sample period is detected it could be estimated using

$$\hat{\rho}_j = c_0 + \gamma t_j, \quad j = 1, \ldots, N.$$ 

For our data set we found the $\gamma$ estimate is not statistically different from zero. However, local trends may be detected and this approach has been tested in Melo.

Fig. 3. PP-plots from the local fit based on the short memory GARCH residuals, and assuming a t-copula with 8 df, $h = 20$, and for the selection of times $0.1, \ldots, 0.9$. 
Local Estimation of Dynamic Copula Models

and Mendes [23], a working paper where the focus is on Value-at-Risk estimation and comparisons.

Motivated by the fact that many empirical studies have reported the increase in dependence during high volatility periods, we now include in the local model the extension where the time variation in the copula parameter is also a time-varying function of the estimated conditional volatilities. According to formulation (3.2), in the case of the t-copula, the correlation coefficient at time $t_j$ is composed by a local constant value $\rho_{0,j}$ plus a dynamic term due to past volatilities.

Of course the final estimates are the same, but through this modeling strategy one can capture (through $\beta_j$) the time-varying impact of volatilities innovations on dependence as well as the time variation in the dependence structure (through $\rho_{0,t}$) due to other factors not affecting volatility. The graphical inspection of the behavior of $\beta_j$ through time would tell us if volatility is indeed an important factor. The graphical inspection of the dynamic variation of $\hat{\rho}_0$ will detect if there are other factors besides volatility affecting the correlation.

To investigate the best functional form for the regressors we plot the series of local estimates ($\hat{\rho}_j$) versus several functions of the GARCH and FIEGARCH volatilities ($\hat{\sigma}_{1,j}, \hat{\sigma}_{2,j}$) and ($\hat{\sigma}_{1,j-1}, \hat{\sigma}_{2,j-1}$), including the average volatility, each one separately, the maximum, and the log of the maximum between the two volatilities, the latter used by Van Den Goorbergh et al. [37]. For the data set under analysis, the plots suggested a weak linear association between the local constant estimates $\hat{\rho}_j$ and the log(max($\hat{\sigma}_{1,j}, \hat{\sigma}_{2,j}$)) and the log(max($\hat{\sigma}_{1,j-1}, \hat{\sigma}_{2,j-1}$)). We found very similar results when using either the GARCH or the FIEGARCH residuals.

Figures 4 and 5 illustrate the results. Figure 4 shows in the upper panel the evolution over time of the estimates of the slope $\beta_j$, and in the lower panel the evolution over time of the log(max($\hat{\sigma}_{1,j}, \hat{\sigma}_{2,j}$)) series from the GARCH fits. We observe that the larger the volatility the greater the slope, thus indicating that volatility does affect the strength of dependence. Figure 5 shows in the upper panel the evolution over time of the final $\hat{\rho}_j$ estimates. In the lower panel we observe the evolution over time of what should be the intrinsic local behavior of the correlation coefficient, due to other factors and beyond the influence of volatilities.

The modeling strategies proposed may be used for prediction and computation of conditional risk measures. To illustrate, we compute the in-sample conditional one-step ahead Value-at-Risk\(^2\) of an equally weighted portfolio based on the SP500 and Nasdaq indexes. To compute the conditional in-sample VaR, returns innovations were sampled from the conditional copula. At each time $t_j$ a total of 100,000 replications were carried on. At the end of the series our model allows for predicting the one-step ahead out-of-sample VaR. The algorithm is given in the Appendix.

Table 2 shows the percentual of times when the VaR estimates underestimated the observed loss. For the sake of completeness we also report the unconditional

\(^2\)The VaR of a portfolio is the value large enough to cover its losses over a N-day holding period with a probability of $(1 - \alpha)$, usually denoted by $VaR(\alpha, N)$, or just $VaR_\alpha$ when $N= 1$. 
Fig. 4. Upper panel: Evolution over time of the estimates of the slope $\beta_j$. Lower panel: the evolution over time of the estimated volatilities $\log(\max(\sigma_1,j,\sigma_2,j))$ from GARCH fits.

Fig. 5. Upper panel: Evolution over time of the final $\hat{\rho}_j$ estimates. Lower panel: the evolution over time of the series $\hat{\rho}_0$. 
Table 2. In sample observed proportion of times the 1% and 5% Value-at-Risk estimates of an equally weighted portfolio failed to estimate the observed loss.

<table>
<thead>
<tr>
<th>Model</th>
<th>1% VaR</th>
<th>5% VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unconditional: weighted portfolio empirical distribution</td>
<td>1.07%</td>
<td>5.06%</td>
</tr>
<tr>
<td>Copula global ρ and marginal GARCH(1,1) one-step ahead predictions</td>
<td>0.77%</td>
<td>5.20%</td>
</tr>
<tr>
<td>Copula local ρ and marginal GARCH(1,1) one-step ahead predictions</td>
<td>0.76%</td>
<td>5.15%</td>
</tr>
<tr>
<td>Copula local ρ and marginal FIEGARCH(1,1) one-step ahead predictions</td>
<td>0.76%</td>
<td>5.14%</td>
</tr>
</tbody>
</table>

5. Dynamic Copula Estimation of Joint Extreme Values

In order to generalize our results we provide now another illustration of local estimation using the Gumbel extreme value copula. The Gumbel copula is a one-parameter (δ ≥ 1) family, where independence corresponds to δ = 1. It is the asymptotic model for componentwise maxima which, in turn, asymptotically follow the generalized extreme value (GEV) distribution.

We start with the decision on the best block size for collecting the minima and maxima. After comprehensive comparisons among the GEV fits based on different block sizes, which included the analysis of the significance of parameters estimates, tests of goodness of fit, qq-plots of the standardized (exponential) residuals, we decided to collect the bi-weekly minima and the bi-weekly maxima. The pseudo uniform (0, 1) data were obtained by plugging in the maximum likelihood estimates in the GEV cumulative distribution function. The plot of the support set for the empirical copula (not shown here) clearly indicates that a Gumbel copula would be adequate for the data.

The $\hat{\delta}$ global maximum likelihood Gumbel copula estimates (standard errors) are respectively 2.6798 (0.1999) and 2.7983 (0.2076) for minima and maxima. They correspond to the (global) tail dependence coefficients of 0.7048 and 0.7189, suggesting symmetric tail dependence for joint minima and maxima. We note that the global (symmetric) tail dependence coefficient based on the global t-copula estimates and daily data is much smaller 0.4888 (correspond to df = 8 and $\rho = 0.8906$).

We also note that the Gumbel $\hat{\delta}$ global maximum likelihood estimates imply in Kendall’s tau estimates of 0.6268 for the minima joint data, and 0.6426 for the maxima joint data. Both Kendall’s estimates differ from the overall Kendall’s $\tau$ estimate of 0.6964 computed from the global maximum likelihood $\rho$ estimate of the t-copula. All these considerations indicate that care is needed when comparing results based on the daily data and maxima (or minima) data. The t-copula with a larger correlation coefficient possess a smaller tail dependence coefficient, while the opposite holds for the Gumbel fit, even though all three Kendall’s $\tau$ correlation coefficient are close.
The choice of the bandwidth $h$ for the local estimation was based on visual assessment of the bias-variance trade off. Recall that the time periods are now half-months, and this would suggest small values for $h$, but we are estimating by maximum likelihood, and the larger the sample, the better. To compromise we considered values between 6 and 15.

Figure 6 shows the evolution over time of the local constant estimates $\hat{\delta}_j$ from the Gumbel copula fit to bi-weekly extremes from SP500 and Nasdaq. The upper panel corresponds to the minima fit and the lower panel corresponds to the maxima fit. The horizontal lines correspond to the global estimates. We observe a slight downward trend in the local estimates. For the minima data the least squares slope estimate (standard error) is $-0.0023(0.0005)$. For the maxima these figures are respectively $-0.0017(0.0004)$. The graphical inspection of the sequence of pp-plots confirmed the good fits.

6. Further Comments and Conclusions

In this paper we applied the local maximum likelihood method for the estimation of copulas parameters. All nice properties of the local estimates extend to the copula
case. The method can be easily implemented and can rapidly show to the practitioner when he/she should rather consider a time-varying copula specification.

Our data come from finance and inference is in two-steps. This is nowadays the classical approach which allows for good marginal fits based on GARCH type models. The copula dependence structure is estimated locally and changes over time. Also, the joint modeling of extremes was considered, and the marginal fits were based on the generalized extreme value distribution.

We discussed the choices of the Kernel function and bandwidth $h$. Expertise information and visual assessment of bias-variance trade-off helps deciding on the best $h$ value. We proposed to assess goodness of fit by graphically examining a sequence of pp-plots. An extention of the local model including the estimated conditional past volatilities as regressors was proposed. The regression coefficient is also time-varying and the results confirmed the well known stylized fact that dependence increases with volatility.

Some issues may be considered data dependent: (i) the choice of $h$ and the sensibility of results to it. This is a practical issue that may affect, for instance, the estimation of risk measures; (ii) the sensibility of results to choice of the Kernel function. This is not crucial and it is a common sense that the Gaussian Kernel does the job. For different applications a heavier tail or asymmetric distribution could be considered.

When working with componentwise maxima there is a chance that the joint data may actually not be observed. However, there is no other good way to collect bivariate minima and maxima. Our objective was just to illustrate the potential of the local estimation method.

Local estimates are useful for more complex models detection. Further modeling may go from simple regression models to more sophisticated ARMA($p,q$) models. Also, extensions based on indicator functions as regressors may be considered. For example, an indicator of bad news may be added to the model, capturing the so called leverage effect.

The empirical analysis carried out in this paper was meant to illustrate the usefulness of models proposed. Of course these results reflect the period covered. Recent joint behavior of financial markets will probably change the numbers obtained here. This is why we need dynamic models. One of the greatest motivations for dynamic estimation of a multivariate distribution comes from finance. The estimated time-varying multivariate distribution may be then used to compute and forecast risk measures, to simulate the behavior of options prices, to select portfolio components, and so on.

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Appendix

Algorithm for Value-at-Risk computation.

For every fixed time \( t, t = 1, \ldots, N - 1 \) do the following:

- At this fixed time \( t \) we estimate the copula correlation coefficient \( \hat{\rho}_{t+1} \) for time \( t + 1 \) using (3.2).
- Simulate 100,000 bivariate observation from a t-copula with 9 df and correlation coefficient \( \hat{\rho}_{t+1} \).
- Find the corresponding quantile-values in the marginal Gaussian distributions (use inverse of normal cdf).
- Using the one-step-ahead predicted GARCH volatility (or FIEGARCH) and the one-step-ahead predicted conditional mean (for each margin), compute the (100,000 bivariate) predicted returns for day \( t + 1 \).
- Form the portfolio value as the average of these two series. The final result is a series of size 100,000.
- Compute the desired VaR as the quantile, the 1% and the 5% quantiles. These are the VaR values for time \( t + 1 \).

References

