Assessing the effect of tail dependence in portfolio allocations

R. P. C. Leal & B. V. M. Mendes

The Coppead Graduate School of Business, The Federal University of Rio de Janeiro, Rio de Janeiro, RJ, 22793-236, Brazil

Institute of Mathematics and The Coppead Graduate School of Business, The Federal University of Rio de Janeiro, Rio de Janeiro, RJ, 22793-236, Brazil

Published online: 25 Jun 2013.

To cite this article: R. P. C. Leal & B. V. M. Mendes (2013): Assessing the effect of tail dependence in portfolio allocations, Applied Financial Economics, 23:15, 1249-1256

To link to this article: http://dx.doi.org/10.1080/09603107.2013.804160

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.tandfonline.com/page/terms-and-conditions

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae, and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand, or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.
Assessing the effect of tail dependence in portfolio allocations

R. P. C. Leal\textsuperscript{a,*} and B. V. M. Mendes\textsuperscript{a,b}

\textsuperscript{a}The Coppead Graduate School of Business, The Federal University of Rio de Janeiro, Rio de Janeiro, RJ, 22793-236, Brazil
\textsuperscript{b}Institute of Mathematics and The Coppead Graduate School of Business, The Federal University of Rio de Janeiro, Rio de Janeiro, RJ, 22793-236, Brazil

Portfolio selection requires an estimate of the degree of association between assets. The Pearson correlation coefficient $\rho$ is the most common measure and estimates the linear correlation implied by the underlying bivariate distribution. Correlations typically rise during stressful times and this nonlinear dependence is measured by a nonzero tail dependence coefficient. We investigate the effect of tail dependence on the estimate of the correlation coefficient. Simulations show that extreme joint losses or gains cause overestimation of the linear correlation coefficient in the presence of tail dependence. The degree of association during the usual days may be smaller than that indicated by the sample correlation coefficient, impacting long-run investments. Simulations show that portfolios based either on the rank correlation or on a conditional version of the Pearson correlation outperform those obtained with classical inputs for moderate and weak strengths of tail dependence association computed from 5 or 10 years of daily data. However, the Pearson correlation coefficient is hard to beat for shorter time horizons and stronger strengths of tail dependence. We recommend estimating the copula pertaining to the data on the presence of tail dependence to select the most suitable correlation coefficient.

\textbf{Keywords:} conditional correlation coefficient; rank correlation coefficient; Markowitz mean-variance model; copulas; tail dependence

\textbf{JEL Classification:} G11; G20

I. Introduction

The mean-variance (MV) model of Markowitz (1952) is the standard tool for portfolio selection. It is a quadratic optimization problem involving maximizing portfolio return while minimizing risk, whose inputs are estimates of the mean vector and the covariance matrix. The traditional difficulty when implementing the MV tool is the choice of these estimates, since the mathematical formulation of efficient portfolios is relatively simple. The widely used inputs are the sample mean and covariance matrix, which possess good properties if the data distribution is actually the multivariate normal. These classical estimates become biased (Hampel et al., 1986) outside the Gaussian world and may lead to unreliable results. Michaud (1989), Michaud (1998), Chan et al. (1999), Ledoit and Wolf (2003), Mendes and Leal (2005) and Okhrin and Schmid (2008), among others, propose alternatives to handle the shortcomings of the classical estimates.

The main drawback of the sample correlation matrix is that it gives equal weights to all data points, resulting in estimates that are very sensitive to extreme outlying

\textsuperscript{*}Corresponding author. E-mail: rleal@ufrj.br

© 2013 Taylor & Francis
observations. A single outlying observation, extreme in just one direction (one portfolio component), for example, may drive the Pearson linear correlation coefficient $\rho$ close to zero. Portfolio optimizations based on new risk measures have been proposed at the cost of simplicity and computational tractability. This is why practitioners usually find it difficult to implement and prefer to apply the MV optimization algorithm based on classical sample estimates or other heuristics (Michaud, 1989).

Markets become more correlated during crises due to contagion (Longin and Solnik, 2001; Embrechts et al., 2002). The strength of correlation on the usual days may be quite different from the value observed during stressful times. The Pearson’s linear correlation coefficient $\rho$, based on the entire data set, may not reflect the strength of interdependence observed during neither low nor high volatility periods.

The amount of dependence between extreme joint quantiles occurring during joint high volatility periods is measured by the tail dependence coefficient $\lambda$. This copula-based nonlinear measure of dependence may be computed for joint gains, the upper tail dependence coefficient $\lambda_U$; and for joint losses, the lower tail coefficient $\lambda_L$.

The tail dependence coefficient is a number between zero and one. It is zero when variables are asymptotically independent. However, tail independence does not mean independence. For example, two portfolio components may possess $\lambda_L$ equal to zero (the the case of Gaussian copula) and $\rho = 0.90$, therefore exhibiting strong correlation during the usual days, but being dissociated during extreme events. On the other hand, two assets may possess Pearson’s $\rho$ equal to zero (or even negative) and a significant positive $\lambda_U$, for instance, $\lambda_U = 0.12$ in the case of a $t$-copula with four degrees of freedom. Thus, $\rho$ equal to zero does not guarantee that there are no co-movements during periods of high volatility.

It is very intuitive that the extreme joint points lying in the corners cause overestimation of the linear correlation coefficient $\hat{\rho}$ when tail dependence does exist. This is another type of contamination suffered by an already nonrobust measure. Consequently, $\hat{\rho}$ cannot be the single number used to reflect the linear correlation implied by the data joint distribution during the majority of days, unless the asset returns are joint normally distributed or independent (and there are no contaminations). The effect of tail dependence on the value of the estimated $\rho$ is assessed in Section IV in a simulation study.

We try to find a simple alternative input to be used in the MV tool, which would result in more stable and meaningful allocations that provide larger gains. This article endeavours to verify if a correlation coefficient that is less affected by extreme joint values provides better allocations for long-run and for short-horizon investments, or frequent rebalancing strategies.

To this end, we run simulations based on large, moderate and small sample historic lengths and consider three correlation coefficients: the empirical estimate of the Pearson correlation coefficient $(\hat{\rho})$, the sample estimate of the Spearman rank correlation $(\tilde{\rho})$ and a conditional version of the sample Pearson correlation coefficient based only on observations within three standard deviations (SDs) ($\rho_C$). We note that this conditional procedure is different from a robust one that ignores or gives lower weights to extreme points. We recognize that extreme points are generated by a different crisis data-generating process, but are not outliers, because they are in the correct direction, despite existing only during extreme volatility days. By considering these points when estimating, one would distort the measure of linear dependence for the vast majority of days.

We go further and compute these three correlation coefficients at three different stages of the whole data-generating process, in order to represent different states of nature: at the observed daily returns space, at the estimated generalized autoregressive conditional heteroscedasticity (GARCH) models standardized residuals environment and at their original innovations space represented by a copula. Statistical tests are carried on to establish the differences among the nine correlation coefficients.

The simulations showed that the Pearson estimate $(\hat{\rho})$ is statistically greater than the rank and the conditional correlations, because it is inflated by extreme values. However, it may provide larger accumulated gains for short horizons (about 2 years) or frequently rebalanced investments, when assets are more strongly associated, and the normal distribution is acceptable. The classical correlation estimates are hard to beat in many plausible investment scenarios.

This article addresses the cases in which the classical correlation matrix is suitable for portfolio optimization problems by means of simulations in a comparison with the rank and a conditional correlation coefficient obtained from tail dependency from copulas. Recent articles claim the superiority of copula-derived tail dependences. For example, DiTraglia and Gerlach (2013) provide evidence that portfolios designed to possess lower tail dependence display a better performance than a market index as well as portfolios with higher tail dependence. They assert that tail dependence is a different risk measure when compared to usual metrics, such as the variance or beta. Boubaker and Sghaier (2013) propose a solution to the portfolio problem by means of minimizing the conditional value at risk with the dependence structure obtained from copulas. In contrast to some of our simulations, they claim that their method outperforms the classical procedure with the linear correlation coefficient representing the dependence structure. Zhou and Gao (2012) assert that the linear correlation coefficient is not adequate in a real estate market linkages application during crises that employs copula tail
dependence metrics. While the literature suggests the superiority of tail dependence metrics, an alternative strain endeavours to provide better estimates of the covariance matrix for portfolio optimization applications. Several authors have pointed out the limitations of the classical covariance matrix, suggesting alternatives to it, such as resampling the efficient frontier (EF), as in Michaud (1998). Huo et al. (2012) use a robust covariance measure based on the median and conclude that robust covariance measures may lead to larger portfolio gains by means of simulations and an out-of-the-sample test. Mendes and Marques (2012) also show, for a particular emerging market, that portfolios estimated with robust pair-copulas based inputs offer greater gains, demand less rebalancing and incur lower trading costs than those that employed the classical correlation.

Shrinkage robust estimators of the covariance matrix are also quite common. Mendes and Leal (2005) offer a robust estimator of the covariance matrix under a shrinkage framework that results from the shrinkage of the classical covariance matrix and a structured and robust estimator of this matrix that does not change the orientation of the data in the presence of extreme values obtained from a spectral decomposition of the original covariance matrix. Ledoit and Wolf (2003) employ the single factor model of Sharpe (1963) as the structured covariance matrix estimator for shrinkage with the classical one. Ledoit and Wolf (2004a) use a constant correlation coefficient in place of the Sharpe (1963) based covariance matrix as the structured covariance matrix estimator while Ledoit and Wolf (2004b) employ an identity matrix as the structured covariance matrix estimator in order to perform their shrinkage. The authors offer their closed-form solution to obtain the shrinkage parameter. Ökhrin and Schmid (2008), for example, compare the classical, book-to-market single-factor, shrinkage and Bayesian estimators of the covariance matrix pointing out, once again, to the advantages of their shrinkage proposal, and offering a solution for the shrinkage parameter.

The literature, thus, is plentiful with suggestions to replace the classical covariance matrix with robust and structured estimators. This article contributes to this literature because it provides the aforementioned conditional version of the sample Pearson correlation coefficient based on extreme observations. Our analysis confirms what the literature states about the effects of extreme values on the classical covariance matrix, but, in contrast, it shows that estimates based on extreme values do not easily beat the classical estimates in any investment horizon or usual asset dependence levels.

The remainder of this article is organized as follows. We briefly review the definition of copulas in Section II and set our arguments in favour of alternate coefficients in Section III. It is important to note that the method applies to high-dimensional data and that the resulting covariance matrices, based on pair-wise correlation coefficients and sample variances, are, by construction, symmetric positive definite matrices. The behaviours of efficient portfolios constructed based on these covariance matrices are compared in a simulation study in Section IV. Section V offers some concluding remarks.

II. Copulas: A Brief Review

We define copulas, for the sake of simplicity, in the bivariate case as well as two important copula-based dependence measures: the rank correlation and the tail dependence coefficients.

Let \( (X_1, X_2) \) be a continuous random variable in \( \mathbb{R}^2 \) with joint cumulative distribution function (CDF) \( F \) and marginal distributions \( G_i, i = 1, 2 \). Suppose that the variables \( X_1 \) and \( X_2 \) are standardized through the probability integral transformation, such that the transformed variables \( U_1 = G_1(X_1) \) and \( U_2 = G_2(X_2) \) are uniformly \((0,1)\) distributed.

The bivariate CDF \( C(\cdot, \cdot) \) of \( (G_1(X_1), G_2(X_2)) \) is the copula of \( (X_1, X_2) \), or the copula pertaining to \( F \). See Joe (1997) and Nelsen (2006) for a comprehensive account of the properties, families, derived dependence measures, and applications of copulas. According to the Sklar (1959) theorem, for all \( (x_1, x_2) \in \mathbb{R}^2 \)

\[
F(x_1, x_2) = C(G_1(x_1), G_2(x_2))
\]

and, in the case where \( G_1(X_1) \) and \( G_2(X_2) \) are continuous, \( C \) is unique. Conversely, if \( C \) is a copula and \( G_1(X_1) \) and \( G_2(X_2) \) are CDFs, the function \( F \) defined in Equation 1 is a joint CDF with marginal distributions \( G_i, i = 1, 2 \).

The copula \( C \) provides all information about the dependence structure of \( (X_1, X_2) \) independently of the specification of their marginal distributions. It is invariant under monotone increasing transformations of \( X_1 \) and/or \( X_2 \), making the copula-based dependence measures interesting scale-free tools for studying dependence. For example, to measure monotone dependence (not necessarily linear) one may use Spearman’s rank correlation \( (r) \) defined as

\[
r(X_1, X_2) = 12 \int_0^1 \int_0^1 u_1 u_2 dC(u_1, u_2) - 3
\]

The rank correlation \( r \) possesses many advantages over Pearson’s correlation \( \rho \): it is invariant under strictly increasing transformations, so it would be the same for the simple and the log-returns; it always exists in the interval \([-1,1]\); it does not depend on the marginal distributions and the values \( \pm 1 \) occur when the variables are functionally dependent, that is, when they are modelled by one of the Fréchet limit copulas (perfect dependence copulas).
The linear correlation $\rho$ is still the preferred measure of association between financial products. Even though $\rho$ is the canonical measure for elliptical distributions, $\rho$ is not a copula-based dependence measure because it also depends on the marginal distributions. In addition to the drawback of measuring only linear correlations, $\rho$ presents other weaknesses, for example, according to Embrechts et al. (2002). We note that $r(X_1, X_2) = \rho(G_1(X_1), G_2(X_2))$, so that the rank and the linear correlations coincide in the copula environment.

The upper tail dependence coefficient $\lambda_U$ between $X_1$ and $X_2$ is defined as

$$
\lambda_U = \lim_{\alpha \to 0} + \lambda_U(\alpha) = \lim_{\alpha \to 0} + \Pr\{X_1 > G_1^{-1}(1 - \alpha) | X_2 \leq G_2^{-1}(1 - \alpha)\}
$$

if this limit exists. The two variables $X_1$ and $X_2$ are said to be asymptotically dependent on the upper tail if $\lambda_U$ is positive, and asymptotically independent if $\lambda_U = 0$. The same concept is applied to define the lower tail dependence coefficient $\lambda_L$.

Both upper and lower tail dependence coefficients may be expressed using the pertaining copula as follows:

$$
\lambda_U = \lim_{u \to 1} \frac{\bar{C}(u, u)}{1 - u}
$$

where $\bar{C}(u, v) = Pr\{U > u, V > v\}$, and

$$
\lambda_L = \lim_{u \to 0} \frac{C(u, u)}{u}
$$

if these limits exist. The measure $\lambda_U$ quantifies the amount of extremal dependence within the class of asymptotically dependent distributions. The tail dependence coefficient is a copula-based measure such as the rank correlation $r$.

For parametric copula families $C_\theta$ indexed by a parameter vector $\theta$, usually, both $\lambda_U$ and $\lambda_L$ are increasing functions of $\theta$. This is very convenient because $\theta$ ranges over its parameter space, some copula families such as the elliptical family are able to model from perfect negative dependence, passing through independence to reach perfect dependence. In the case of extreme value copulas, that $\theta$ ranges over its parameter space, the copula models from independence to perfect positive dependence. Copulas for modelling joint returns may be the product copula, the copula of independent marginals, or some member of the elliptical family, such as the Gaussian and the $t$-copula. These choices are supported by theoretical and empirical studies.

The two parameters $t$-copula, an important member of the elliptical copulas, is given by

$$
C_t^\alpha(u_1, u_2) = \int_{-\infty}^{t^{-1}u_1} \int_{-\infty}^{t^{-1}u_2} \frac{1}{2\pi(1 - \rho^2)^{1/2}} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \left( \frac{s^2 - 2\rho st + t^2}{\nu(1 - \rho^2)} \right)^{\nu/2} \right\} \, ds \, dt
$$

where $t^{-1}(\cdot)$ and $t(\cdot)$ represent, respectively, the quantile function and the CDF of a univariate standard $t$-distribution, $t_\nu$, with $\nu$ degrees of freedom. This is the copula pertaining to a bivariate $t$-distribution ($t^\nu_2$) with $\nu$ degrees of freedom and correlation coefficient $\rho$. We note that the number of degrees of freedom $\nu$ is the same for the copula and margins of the $t^\nu_2$. Likewise all elliptical copulas, the correlation coefficient in the copula space is slightly smaller than the $\rho$ of the bivariate $t$-distribution.

The $t$-copula density, $c_t^\nu$, is given by

$$
c_t^\nu(u_1, u_2) = \left| \begin{array}{c}
\rho^{\nu/2} \Gamma\left(\frac{\nu + 1}{2}\right) \\
\Gamma\left(\frac{\nu}{2}\right) \prod_{i=1}^{2} \left(1 + \frac{u_i^2}{\nu}\right)^{-(\nu + 1)/2}
\end{array} \right|^{1/2}
$$

where $\rho$ is a $2 \times 2$ matrix of correlations with diagonal elements equal to one, and the $\rho$ values outside the diagonal, and where $q_i = t^{-1}(u_i)$.

Parametric estimation of copulas is usually accomplished in two steps according to the inference functions for margins (IFMs) method (Joe, 1997). In the first step, conditional or unconditional models are fitted to each margin, and the pseudo uniform $(0,1)$ data are obtained through the probability integral transformation. The transformed data are used in the second step to estimate the best parametric copula family. The goodness of fits may be assessed visually through $pp$-plots or based on some formal goodness-of-fit test, usually based on the empirical copula and intending to minimize a distance criterion. For a discussion on goodness-of-fit tests, see Genest et al. (2009).

### III. Incorporating Tail Dependence to the Markowitz’s MV Model

We begin by discussing how daily returns are generated. Figure 1 schematically represents the return data-generating process.

Suppose we are working with daily returns from $d$ risk factors, $r = (r_1, r_2, \ldots, r_d)$, covering a period of length $T$. On every business day $t$, $t = 1, 2, \ldots, T$, nature generates a $d$-dimensional vector of errors $u = (u_1, u_2, \ldots, u_d)$ marginally distributed according to the standard uniform distribution, and linked by a dependence structure (the oval box in Fig. 1) establishing all relationships between the
marginal errors, which might have been caused by economic, political and geographic factors, besides some market macro-structure factors. This is stage 1. Note that all information about the existence of linear association and tail dependence are set at this stage.

Then the errors vector passes through a black box holding information on the underlying unconditional marginal distributions $F_1, F_2, \ldots, F_d$, not necessarily the same, possibly asymmetric and heavy tailed, all possessing zero mean and unit variance. Information in this stage (again, caused mainly by economic, political and geographic factors) changes the value of the linear correlation coefficient but does not change the value of the rank correlation. The output from this box is $z = (z_1, z_2, \ldots, z_d) = (F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_d^{-1}(u_d))$. This is stage 2.

Finally, the $z$ vector passes through a squared black box holding information on market micro-structure, occasional bad news, seasonalities and so on. The output from this squared box is the daily observed returns $(r_1, r_2, \ldots, r_d)$, which usually possess serial correlations, conditional heteroscedasticity and show some extreme values. This is stage 3. Note that at this stage there are linear and nonlinear transformations, not necessarily strictly increasing, which will affect all moments, in particular the mean vector, $\mu$ and all dependence measures. At the end of the period $T$, one has the historical $T \times d$ return data.

In practice, in order to obtain the inputs for the MV model, the analyst needs first to identify the mechanism inside the three boxes, and estimation is performed in the reverse order. Using the currently available computing facilities, nowadays it is possible to obtain excellent univariate (conditional and unconditional) fits, tailored for each returns series. The conditional fits, based on some autoregressive fractionally integrated moving average (ARFIMA) $(p, d, q)$ combined with a fractionally integrated exponential GARCH (FIEGARCH) $(r, d, s)$ type model, are able to capture the temporal dynamics in the mean and in the volatility (see Crato, 1994; Poon et al., 2003; Mendes and Kolev, 2008). Actually, many empirical works have shown that a simple GARCH (1,1) model with a leverage term, and therefore it possesses the same specification of a threshold GARCH (TGARCH) model with a threshold value of zero, and innovations $z$, following a $t$-student distribution is able to capture all temporal dynamics in the data (Hansen and Lunde, 2005). This is why we only consider this simple model in the simulations in Section IV. From the $d$ models fitted (say, $G_1, G_2, \ldots, G_d$), the analyst obtains $d$ series of standardized (zero mean and unit variance) filtered data of stage $2, (z_1, z_2, \ldots, z_d)$ for each day $t$, and identify their marginal distributions $(\tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_d)$.

The support set for the empirical copula is the transformed data in stage 3, $(u_1, u_2, \ldots, u_d) = (\tilde{F}_1(z_1), \tilde{F}_2(z_2), \ldots, \tilde{F}_d(z_d))$. They are used to identify the dependence structure (a copula or a pair-copula) represented by the oval box.

At this point, the analyst could estimate the dependence structure by selecting appropriate $d$-dimensional copula families and testing for the best fit. However, here our problem is much simpler. We just need to find the best bivariate copula fit for each $(i, j)$ pair of transformed data $(u_i, u_j), i \neq j, i, j = 1, \ldots, d$. The purpose is just to identify if there is tail dependence. If $\lambda_U$ or $\lambda_L$ is positive for $d$-1 leading pairs, then there is a copula decomposition (a pair-copula) possessing multivariate tail dependence (Joe et al., 2010). This information is used to select the most suitable estimate of the correlation coefficient, indicated by the simulation results in Section IV.

Sample estimates of the three correlation coefficients $\hat{p}_{ij}(S), \hat{r}_{ij}(S), \text{ and } \hat{p}_{ij}(S), \text{ for } i \neq j, i, j = 1, \ldots, d$ are computed at the three stages ($S$), $S = 3, 2, 1$, using, respectively, the observed data $r$, the estimated data $z$ and the estimated and transformed data $u$. The final correlation estimate to be used as the MV input will be selected based on simple rules derived from the simulations in Section IV, which also take into account a few additional variables: the series length, rebalancing policy and investment risk.

To complete the MV tool specification, we take the classical sample means $\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_d$, and compute the $d \times d$ covariance matrices using the sample variances $\hat{\sigma}_1^2, \hat{\sigma}_2^2, \ldots, \hat{\sigma}_d^2$ and the selected correlation coefficients. It is important to note that all high-dimensional covariance matrices obtained are based on pair-wise correlation coefficients and are, by construction, symmetric positive definite matrices.

IV. Simulations

We carried out simulation experiments, which will lead to the recommendations at the end of this section, to verify if a correlation coefficient that is less affected by extreme
joint values provides better allocations for long-run and for short-horizon investments, or frequent rebalancing strategies. We considered long (‘10 years’, 2640 business days), intermediate (‘5 years’, 1320 business days) and short (‘2 years’, 528 business days) sample sizes (SS), or time series length, representing a long, intermediate and short-horizon investments. We are aware that the number of business days in the year is approximately 252, therefore, we use ‘10 years’, ‘5 years’ and ‘2 years’ in quotes as the approximate size of our samples.

We considered two members of the class of elliptical distributions F to generate the data: the normal and \( t_6 \) bivariate distributions. This means that the marginal distributions \( F_1 \) and \( F_2 \) are either the standard normal or the \( t \)-student with six degrees of freedom. The values for the correlation \( \rho \) were chosen as 0.10, 0.30 and 0.60, representing weak, moderate and strong strengths of association, respectively. The number of repetitions (\( N \)) was 10 000. The GARCH (1,1) model chosen has assumed values for the parameters suggested by empirical applications: \( a_0 = 0.10, \alpha_1 = 0.10, \beta_1 = 0.88, \) and leverage term equal to \( -0.15. \) For example, the value of the \( a_0 \) parameter seemed to be a reasonable input given its estimates for a leveraged GARCH (1,1) for the daily returns of the Morgan Stanley Capital International (MSCI) indices of some emerging markets, which resulted in 0.136 for Brazil, 0.054 for China, 0.129 for Indonesia and 0.093 for Russia.

The simulation procedure is as follows, for \( j = 1, ..., N \):

1. Generate data \( u \) of size \( SS \times 2 \) from the bivariate copula pertaining to \( F \).
2. Obtain the data \( z \) applying the quantile functions \( F_1^{-1} \) and \( F_2^{-1} \) to \( u \).
3. Simulate from the specified GARCH model with errors data \( z \), obtaining \( r \). Using \( r \), compute the three correlation coefficients at this stage, \( \hat{\rho}(3), \hat{r}(3) \) and \( \hat{\rho}_C(3) \).
4. Assuming that \( r \) are the observed returns data, start the real-life estimation process by estimating the conditional GARCH models, \( \hat{G}_1 \) and \( \hat{G}_2 \). From the GARCH fits, obtain the standardized residuals \( \hat{z} \) and compute the three correlation coefficients at this stage, \( \hat{\rho}(2), \hat{r}(2) \) and \( \hat{\rho}_C(2) \).
5. Through the CDFs \( \hat{F}_1 \) and \( \hat{F}_2 \), obtain the copula data \( \hat{u} \) and compute the three correlation coefficients at this stage, \( \hat{\rho}(1), \hat{r}(1) \) and \( \hat{\rho}_C(1) \).
6. Fit copulas to data \( \hat{u} \) and test for the existence of tail dependence. Save this information.
7. Take as MV inputs, the sample mean and the sample SDs of the returns data \( r \) and the covariance matrices based on all nine correlation estimates, obtaining the EFs at each one of the three stages. We choose three optimal portfolios to be compared with respect to cumulative gains in each EF: (i) the minimum variance (MinRisk); (ii) the maximum return (MaxRet); (iii) the middle portfolio (Middle), whose expected return is half way from the other two, and compute measures of portfolio performance.

We initially compare the values of the correlation coefficients within each stage. We carry on \( t \)-tests at the 5% significance level, wherein the alternative hypothesis is that the linear correlation \( \rho \) at stage \( S, S = 1, 2, 3 \), is affected by the presence of tail dependence, being greater than the rank and greater than the conditional correlation coefficients in the same stage. In addition, since \( \hat{\rho}(3) \) is the estimate widely used in applications, we also test if \( \rho(3) \neq \hat{r}(2) \) and if \( \rho(3) \neq \hat{\rho}_C(2) \).

Under the \( r \)-distribution, the tests strongly indicate that \( \hat{r}(S) \) and \( \hat{\rho}_C(S) \) are statistically smaller than the classical \( \hat{\rho}(S) \), for all \( S = 1, 2, 3 \). In Table 1, we provide the \( p \)-values for stage 2, and for the additional test comparing stages 2 and 3, all based on intermediate, ‘5 years’, sample sizes (results did not change for the other sample lengths). Under the normal distribution, typically the estimates were statistically indistinguishable.

To assess the best correlation choice in applications, for each simulation and portfolio type, and within each stage \( S \), we compute the proportion of days \( A \% \) in which the accumulated gains from the alternative portfolios (based on either \( \hat{r}(S) \) or \( \hat{\rho}_C(S) \)) were greater than the

<table>
<thead>
<tr>
<th>( H_1 )</th>
<th>( \rho(2) &gt; \hat{r}(2) )</th>
<th>( \rho(2) &gt; \hat{\rho}_C(2) )</th>
<th>( \rho(3) \neq \hat{r}(2) )</th>
<th>( \rho(3) \neq \hat{\rho}_C(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>Bi-variate ( t ) distribution (( v = 6 ))</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>0.60</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>0.30</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>0.10</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Assessing the effect of tail dependence in portfolio allocations

Table 2. Average proportion of days (A%) yielding higher accumulated gains for the rank based \( \hat{r}(S) \) and the \( \hat{\rho}_C(S) \) compared to the classical version \( \rho(S) \) for portfolios computed from long ('10 years') times series

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>Correlation coefficient</th>
<th>( \rho = 0.10 )</th>
<th>( \lambda_U = 0.022 )</th>
<th>( \rho = 0.30 )</th>
<th>( \lambda_U = 0.093 )</th>
<th>( \rho = 0.60 )</th>
<th>( \lambda_U = 0.230 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multivariate t-distribution</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MinRisk</td>
<td>( \hat{r}(2) \times \rho(3) )</td>
<td>73.33 (8.65)</td>
<td>65.39 (7.29)</td>
<td>61.94 (6.37)</td>
<td>( \rho_C(2) \times \rho(3) )</td>
<td>74.02 (8.44)</td>
<td>66.99 (7.27)</td>
</tr>
<tr>
<td>MinRisk</td>
<td>( \hat{r}(2) \times \rho(2) )</td>
<td>72.14 (8.78)</td>
<td>70.27 (7.88)</td>
<td>62.73 (6.16)</td>
<td>( \rho_C(2) \times \rho(2) )</td>
<td>60.83 (8.61)</td>
<td>59.16 (7.88)</td>
</tr>
<tr>
<td>MinRisk</td>
<td>( \hat{r}(2) \times \rho(3) )</td>
<td>78.21 (9.52)</td>
<td>63.24 (9.78)</td>
<td>63.71 (8.02)</td>
<td>( \rho_C(2) \times \rho(3) )</td>
<td>77.74 (9.95)</td>
<td>65.39 (9.82)</td>
</tr>
<tr>
<td>Middle</td>
<td>( \hat{r}(2) \times \rho(2) )</td>
<td>71.76 (9.89)</td>
<td>68.15 (9.55)</td>
<td>64.13 (8.16)</td>
<td>( \rho_C(2) \times \rho(2) )</td>
<td>60.31 (9.67)</td>
<td>61.72 (9.60)</td>
</tr>
<tr>
<td>Gaussian copula</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MinRisk</td>
<td>( \hat{r}(2) \times \rho(3) )</td>
<td>53.01 (8.16)</td>
<td>48.69 (5.78)</td>
<td>48.35 (3.49)</td>
<td>( \rho_C(2) \times \rho(3) )</td>
<td>52.50 (8.20)</td>
<td>49.35 (5.82)</td>
</tr>
<tr>
<td>MinRisk</td>
<td>( \hat{r}(2) \times \rho(2) )</td>
<td>47.14 (8.18)</td>
<td>51.45 (5.55)</td>
<td>55.70 (4.18)</td>
<td>( \rho_C(2) \times \rho(2) )</td>
<td>48.73 (8.27)</td>
<td>51.20 (5.60)</td>
</tr>
<tr>
<td>MinRisk</td>
<td>( \hat{r}(2) \times \rho(3) )</td>
<td>54.22 (9.15)</td>
<td>50.05 (8.28)</td>
<td>48.32 (6.40)</td>
<td>( \rho_C(2) \times \rho(3) )</td>
<td>53.74 (9.60)</td>
<td>51.07 (8.27)</td>
</tr>
<tr>
<td>Middle</td>
<td>( \hat{r}(2) \times \rho(2) )</td>
<td>46.76 (9.65)</td>
<td>50.23 (8.78)</td>
<td>53.74 (6.38)</td>
<td>( \rho_C(2) \times \rho(2) )</td>
<td>47.31 (9.70)</td>
<td>48.59 (8.88)</td>
</tr>
</tbody>
</table>

Accumulated gains from the classical version based on \( \rho(S) \). We also compare the performance of portfolios based on \( \hat{r}(2) \) or \( \hat{\rho}_C(2) \), since we estimated the true multivariate distribution in stage 2, to those based on \( \rho(3) \) from stage 3 where the correlation coefficient is usually estimated. We report the average of these A% values and their SEs in Table 2.

Table 2 summarizes the results obtained for long samples. The results for intermediate sample sizes are very similar, whereas, for short samples, most of the portfolios constructed were statistically equivalent. We also do not report the results for the maximum return portfolio because there were no significant differences between methods.

Despite the strong results in Table 1, the consequences carried on to the portfolios are not so dramatic. This was expected because the most important input to the MV algorithm is the estimate of the mean vector, which is the same when computing all EFs in our simulations. Even though, for long and intermediate sample sizes, the rank estimates provided portfolio compositions yielding higher accumulated gains than those based on the standard method.

We observe that the linear correlation computed in stage 3 (returns) may not be the best correlation coefficient for the MV tool when the data possess heavy tails linked through nonlinear forms of dependence. Although the variability is high, there is a gain when constructing the covariance matrices based on either the rank or the conditional correlation, at least for moderate and weak strengths of association. Due to the nonlinear transformations implied by the GARCH models, it seems that correlation estimates may result in higher accumulated gains when computed at the level of the multivariate underlying distribution, stage 2 of the standardized GARCH residuals.

V. Concluding Remarks

In this article, we proposed a procedure for incorporating the information found in the joint tails of the data when estimating the correlation coefficient \( \rho \), aiming at financial applications. We chose the MV optimization method for portfolio allocations to illustrate our methods. After fitting copula models to the data, we identified if there was tail dependence and used this information to select the appropriate correlation coefficient.

Simulations and the statistical tests carried out provided strong evidence to support the claim that \( \rho \) is more affected by the dependence at the tails and is greater than other measures of association. Our evidence suggests that whenever there is tail dependence, the analyst could obtain optimal portfolios with better long-run performance by applying either rank or the conditional correlation coefficients.

In summary, we can say that the rank and the conditional correlation coefficients will work better than
the classical one when there is tail dependence for long (about 10 years) and intermediate (about 5 years) sample sizes and weak (0.10) through moderate (0.30) strengths of association. However, in cases of smaller sample sizes (about 2 years), greater strengths of association (0.60), and under the normal distribution, these estimates are as good as the classical one, which is hard to beat.

References


